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EXPLOITING ADMITTANCE FORMALISM IN THE NONLINEAR ANALYSIS

Summary – The fundamental issues and ideas regarding the use of the admittance formalism in the frequency domain analysis of weakly nonlinear circuits are addressed in this paper. We begin with presenting and discussing in detail the time domain basics important for this analysis. These are the constitutive equations of basic nonlinear circuit elements and the Volterra series. The need for distinguishing between the input-output and in-network type descriptions for nonlinear circuit elements is pointed out. This topic is illustrated by derivations of the aforementioned description types for some two-terminal nonlinear circuit elements devoted to the use in descriptions of weakly nonlinear networks with a single input port. The notion of nonlinear admittance, introduced in one of the recently published papers, is discussed in the context of the modified nodal formulation. The latter uses the modified admittance matrix. Finally, the restrictions regarding a certain operator introduced in the literature to simplify the calculus are pointed out.

1 Introduction

Till appearance of the paper [1], the notion of a nonlinear admittance was not used in the literature on the analysis of weakly nonlinear circuits. In [2], being a milestone paper in the area, the authors speak only about the admittance matrix regarding the linearized part of a mildly nonlinear network. They evaluate the influence of nonlinearities of the corresponding orders on the network behavior in an iterative process using this matrix and a kind of independent sources. The latter are assumed to be independent at a given iteration step, however, their parameters are calculated using the results obtained in the previous steps. Moreover, the values of the coefficients of power series expansions, in which the relationships between terminal variables of basic nonlinear circuit elements are developed, are used in their evaluation. So no need for introducing the notion of a nonlinear admittance in such an approach appears. Hence, a question arises what we really get more, with respect to the old theory, when we introduce and use this notion, as in [1], analyzing weakly nonlinear circuits.

The objective of this paper is to discuss the fundamentals of the approaches presented in [1] and [2], and of the tools derived and used therein. Another way to express it is that we put these means of analysis into a more general framework. And to this end, we use such analytical tools as the constitutive equations of basic nonlinear circuit elements, elements of the theory of polynomial mappings, and others.

The main results achieved in this paper are the following:

A general framework in the frequency domain for evaluation of the partial responses of a weakly nonlinear network, for the corresponding orders of nonlinearities incorporated in it, has been formulated. This approach is more general than the probing method [2] or the means of calculation presented in [1], [4] because it does work independently of the form of an excitation.

Moreover, it has been shown that in the nonlinear analysis of mildly nonlinear networks we must take care of the context. That is, more precisely, depending upon the place where the input signal is applied to a network, one must choose between the input-output or in-network type description for a nonlinear element incorporated in it. The expressions for these descriptions in a general framework have been derived.

2 Constitutive equations for nonlinear conductor and capacitor

In a very seminal paper [3] on electronic device modeling, Chua defines four basic two-terminal circuit elements: resistor, inductor, capacitor, and memristor through the so-called constitutive relations. That is through some algebraic relationships between voltage and current, between flux and current, between charge and voltage, and finally between flux and charge, respectively, where the quantities listed are assumed to be waveforms (signals) of a continuous time *t*. So, for example, for a resistor, we write (using the same notation as in [3])

$$f_R\left(v,i;t\right)=0 \tag{1a}$$

and

$$f_R\left(v\left(t\right),i\left(t\right)\right)=0 \;\text{ or }\; f_R\left(v,i\right)=0 \tag{1b}$$

in the time-invariant case. In (1a) and (1b), *v* and *i* mean, respectively, the voltage (across a resistor) and the current (flowing through this resistor). Moreover, f_R stands for an algebraic relationship that is a one involving only strictly algebraic operations (for more details, see [3]). In usual case, it is a function, generally, a nonlinear one. So, in the latter case, we say that it describes a nonlinear resistor. When however, it reduces to a linear function going through the point $\{v=0,\ i=0\}$, we speak about a (strictly) linear resistor. Furthermore, observe that (1a) or (1b) define implicitly the relationship existing between the voltage and current at the resistor.

This means of description of basic circuit elements outlined above has many advantages. It is axiomatic, general, and didactical, therefore, best suited to provide a good starting point for our further considerations.

In what follows, we restrict ourselves to the time-invariant case. That is we refer to (1b). And solving (1b) for *v* as a function of *i* provides an explicit relationship between the terminal variables (*v* and *i*) of a resistor. Observe that (1b) can be also solved in another way, for getting *i* as a function of *v*. Then, we say, alternatively, that we arrived at an explicit relationship for a conductor (that is for a resistor named now a conductor).

We will use, in what follows, the latter form of the explicit relationship between the terminal variables of a resistor (called now a conductor). And denote it by

$$i=i(v) \tag{2}$$

showing that *i* is a function of *v*, generally, a nonlinear one. Moreover, assume that the conductor considered is a part of a network (circuit) in which the currents and voltages consist of two components: the quiescent (dc) ones and alternating (ac) ones. In this case, we say that the first of them determine the so-called operating points at the circuit elements (or devices) and the second ones are the so-called small signal parts of the currents and voltages.

For this environment, the function (2) can be expanded around the conductor operating point in a Taylor series as

$$i(v)=i(V)+\left.\frac{di}{dv}\right|_{v=V}\cdot(v-V)+\frac{1}{2!}\left.\frac{d^2i}{dv^2}\right|_{v=V}(v-V)^2+\frac{1}{3!}\left.\frac{d^3i}{dv^3}\right|_{v=V}(v-V)^3+\ldots \tag{3a}$$

where $i(V)=I$ and V mean the dc components of current and voltage, respectively, at the conductor. Moreover, the derivatives in (3a) of the first, second, and third order, and so on, of $i=i(v)$ are calculated at the operating point (represented by the voltage *V*).

Note that (3a) can be rewritten in the following way

$$i_\Delta = i\left(V + v_\Delta\right) - i\left(V\right) = \left.\frac{di}{dv}\right|_{v=V} v_\Delta + \frac{1}{2!}\left.\frac{d^2 i}{dv^2}\right|_{v=V} {v_\Delta}^2 + \frac{1}{3!}\left.\frac{d^3 i}{dv^3}\right|_{v=V} {v_\Delta}^3 + \ldots \quad (3b)$$

where $i_\Delta = \Delta i$ and $v_\Delta = \Delta v$ stand for the ac components. Moreover, it follows from (3a) and (3b) that the following notation: $i = i\left(v\right) = i\left(V + v_\Delta\right) = i\left(V\right) + i_\Delta$ and $v = V + v_\Delta$ is exploited therein.

Observe now that for $v = V$ we get from (3a)

$$i\left(V\right) = I \;\; \rightarrow \;\; i\left(V\right) - I = f_R\left(V, I\right) = 0 \quad (4a)$$

and from (3b)

$$i_\Delta - \left.\frac{di}{dv}\right|_{v=V} \cdot v_\Delta - \frac{1}{2!}\left.\frac{d^2 i}{dv^2}\right|_{v=V} \cdot {v_\Delta}^2 - \frac{1}{3!}\left.\frac{d^3 i}{dv^3}\right|_{v=V} \cdot {v_\Delta}^3 + \ldots = f_r\left(v_\Delta, i_\Delta\right) = 0 \;. \quad (4b)$$

The interpretation of (4a) and (4b) is such that in this environment of "an operating point plus a small signal" we have two separate constitutive equations: one for the operating point – given by (4a) – relating to each other its dc terminal variables, and the second for the small signal part – given by (4b) – relating to each other the conductor ac terminal variables. Moreover note that when $f_R\left(\cdot,\cdot\right)$ is a nonlinear function, its form differs, generally, from the form of the nonlinear function $f_r\left(\cdot,\cdot\right)$ defined by (4b). Furthermore, one uses the constitutive equation (4a) in the dc analysis of a circuit, but the constitutive equation (4b) in its small signal analysis.

Conversely, it is interesting to see that using these results one can define what was named above an "an operating point plus a small signal" environment. To this end, we apply (4a) and (4b) in (3a) what gives

$$f_R\left(V + v_\Delta, I + i_\Delta\right) = f_R\left(V, I\right) + f_r\left(v_\Delta, i_\Delta\right) = 0 \quad (5)$$

with the constitutive equations $f_R\left(V, I\right) = 0$ and $f_r\left(v_\Delta, i_\Delta\right) = 0$ fulfilled, each of them, separately. Further, consider a circuit consisting of a connection of nonlinear resistors and assume that the excitations in it are the dc and ac ones, applied at the same time. Then, if each constitutive equation of the resistors in the circuit can be partitioned as in (5) with $f_R\left(V, I\right) = 0$

and $f_r(v_\Delta, i_\Delta) = 0$, we say about this circuit that it works under "an operating point plus a small signal" regime.

In the context of derivations of this section, it is worthy to note that (4b) can be also viewed as a Maclaurin expansion of the function

$$i_\Delta = i_\Delta(v_\Delta) = i(V + v_\Delta) - I = i(V + v_\Delta) - i(V) \tag{6a}$$

where *V* and *I* are some constants. That is it is, after definition, the expansion of (6a) in form of a Taylor series around the point $v_\Delta = 0$.

This is so because

$$\left.\frac{di_\Delta}{dv_\Delta}\right|_{v_\Delta=0} = \left.\frac{di}{dv}\right|_{v=V}, \quad \left.\frac{d^2 i_\Delta}{dv_\Delta^2}\right|_{v_\Delta=0} = \left.\frac{d^2 i}{dv^2}\right|_{v=V}, \text{ and so on.} \tag{6b}$$

Hence, applying (6b) on the left-hand side of (4b) gives

$$i_\Delta = \left.\frac{di_\Delta}{dv_\Delta}\right|_{v_\Delta=0} \cdot v_\Delta + \frac{1}{2!}\left.\frac{d^2 i_\Delta}{dv_\Delta^2}\right|_{v_\Delta=0} \cdot {v_\Delta}^2 + \frac{1}{3!}\left.\frac{d^3 i_\Delta}{dv_\Delta^3}\right|_{v_\Delta=0} \cdot {v_\Delta}^3 + \ldots \tag{7}$$

which is really the Maclaurin expansion of (6a).

In what follows, we will analyze only the ac signals (ac parts of terminal variables). Therefore, for example for nonlinear resistors, we will prefer to use the description in form of (7). Further, (7) can be rewritten in a more convenient form of a power expansion as

$$i_\Delta = g_1 v_\Delta + g_2 v_\Delta^2 + g_3 v_\Delta^3 + \ldots . \tag{8}$$

The corresponding formulas for the coefficients g_1, g_2, g_3, and so on in (8) are evident by comparison of (8) with (7).

At this point, it is also worthy to mention the following. In [3], the right pairs of waveforms $(V + v_\Delta, I + i_\Delta)$ and (V, I) in (1b) are called the admissible voltage-current signal pairs associated with the conductor (resistor) considered. Moreover, the relation (1b) can be viewed as its "natural" constitutive equation. Therefore, for a given operating point (V, I), each pair (v_Δ, i_Δ) that is a right ac part of $(V + v_\Delta, I + i_\Delta)$ can be considered as an admissible voltage-current ac signal pair associated with the resistor "natural" constitutive equation (1b). However, this pair is not necessarily an admissible pair in the strictly small signal constitutive equation (4b) of the resistor, too. This can happen, for example, as a result of lack of convergence of the power series in (4b) for some set of pairs (v_Δ, i_Δ) for which $f_R(V+v_\Delta, I+i_\Delta)=0$ is still valid.

Consider now the next two-terminal basic element, a capacitor. According to [3], its constitutive equation for the time-invariant case has the following form

$$f_C\left(q(t), v(t)\right)=0 \;\text{ or }\; f_C(q,v)=0 \tag{9}$$

where *q* and *v* mean, respectively, the charge (gathered on a capacitor) and the voltage (across a capacitor). Moreover, f_C stands for an algebraic relationship that is a one involving only strictly algebraic operations (for more details, see [3]).

Equation (9) constitutes an implicit relationship existing between the charge and voltage at the capacitor. Solving it for *q* as a function of *v* provides an explicit relationship between the terminal variables (*q* and *v*) of a capacitor. We denote this function by

$$q = q(v) \tag{10}$$

which is assumed here to be nonlinear.

Applying the same procedure that was used before in the case of the nonlinear resistor to (10), we get the counterpart of (8) in the form

$$q_\Delta = c_1 v_\Delta + c_2 v_\Delta^2 + c_3 v_\Delta^3 + ... \;. \tag{11a}$$

where now $q_\Delta = \Delta q$ and $v_\Delta = \Delta v$ stand for the ac components of the charge and voltage, respectively, at the capacitor. And the coefficients $c_1, c_2, c_3, ...$ in (11) are given by

$$c_1 = \left.\frac{dq}{dv}\right|_{v=V},\; c_2 = \frac{1}{2!}\left.\frac{d^2 i}{dv^2}\right|_{v=V},\; c_3 = \frac{1}{3!}\left.\frac{d^3 i}{dv^3}\right|_{v=V} \;... \tag{11b}$$

where *V* means the quiescent voltage at the capacitor. Moreover, similarly as before, (11a) can be interpreted as the Maclaurin expansion of the function

$$q_\Delta = q_\Delta(v_\Delta) = q(V+v_\Delta) - Q = q(V+v_\Delta) - q(V) \tag{11c}$$

that is the Taylor series expansion around the point $v_\Delta = 0$. *Q* in (11c) denotes the quiescent charge at the capacitor.

Note that one terminal variable in (11a) is neither a voltage nor current. To have the (implicit) form of the constitutive equation of the nonlinear capacitor for ac components as that given by (11a), which however possesses exclusively the terminal variables being voltages or currents, we must use an additional equation

$$i_{\Delta}(t)=\frac{dq_{\Delta}(t)}{dt} \text{ or shortly } i_{\Delta}=\frac{dq_{\Delta}}{dt} . \tag{12}$$

So introducing (11a) into (12) gives

$$i_{\Delta}=\frac{d}{dt}\left(c_1 v_{\Delta}+c_2 v_{\Delta}^2+c_3 v_{\Delta}^3+....\right). \tag{13}$$

Dropping the index "Δ" everywhere in (8) and (13) for simplicity of notation, we get

$$i=g_1 v+g_2 v^2+g_3 v^3+.... \tag{14a}$$

and

$$i=\frac{d}{dt}\left(c_1 v+c_2 v^2+c_3 v^3+....\right) \tag{14b}$$

where now *i* and *v* mean exclusively the ac parts of the terminal variables at the conductor or at the capacitor, respectively.

The right-hand side of (14a) and the expression under the operation of differentiation in (14b) represent infinite power series. So it can happen that they will not converge for some values of *v*. The radii of convergence of these series can be calculated using, for example, Cauchy criterion. We get then

$$|v|<\frac{1}{\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|g_n|}}=r_{cg} \text{ and } |v|<\frac{1}{\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|c_n|}}=r_{cc} \tag{15}$$

where r_{cg} and r_{cc} mean the corresponding radii of convergence of the expansion for a conductor and for a capacitor, respectively.

We draw the reader's attention to the fact that the restrictions of the type shown in (15) do not occur in cases of the linear resistors and capacitors as well as of such the nonlinear resistors and capacitors for which the series as those occurring in (14a) and (14b) but with only a finite number of components constitute their exact descriptions. (At the same time, note however that the assumption of exactness of the description by a truncated series has no value in practice. This is so because such the series is used in modeling as an approximate expansion, valid, approximately, for only a bounded set of signals, what excludes possessing the aforementioned property.)

Finally, note that the other two basic nonlinear two-terminal elements: inductor and memristor, and all the more complicated elements [3] as well, can be viewed and analyzed in the time domain similarly as the two-terminal nonlinear resistor and capacitor dealt with above. Detailed study of this type is however beyond the scope of this paper.

3 Input-output and in-network type descriptions of circuit elements

In fact, in [1], [4], and [5], two types of descriptions of basic and of larger circuit elements (modules) as, for example, weakly nonlinear amplifier blocks were used. A trial to unify them sometimes led to confusing results.

Here, we use the following terminology for the above descriptions: of input-output type and of in-network type. For both of them, the basic equation is the same and it is their implicit constitutive equation (such as (14a) or (14b) discussed in the previous section). However, they differ from each other in that:

In the input-output type description, one terminal variable is its input signal, and the second one is its output signal.

In the in-network type description, none of the terminal variables is its input signal. That is in this case the input signal is applied to a place different from the element's port.

In the above definitions, we restricted ourselves, not to complicate too much, to consideration of only two-terminal elements (what suffices for the purposes of this paper).

Once again, we draw the reader's attention to the fact that in spite of having the same constitutive equation the aforementioned description (models) are different. The difference between them is determined by the place of applying the input signal (directly to the element or not) and the state of being a stand-alone element (in the case of the input-output type description) or an in-side element of a network (consisting of elements connected in some way with each other). So the contexts of these models (descriptions) are different what causes that we can not identify them. Shortly, these models differ by their contexts.

Beside the same constitutive equation, these descriptions possess one common feature more. This is their iterative character, which we will explain in course of the derivations of this section. For explanations here, we will use the basic nonlinear elements described in detail previously that is the nonlinear conductor and capacitor.

So assuming now, for example, in the implicit constitutive equation of the conductor given by (14a) the terminal variable $v = v_i$ to be the input signal and $i = i_o$ to be, accordingly, the output signal, we get for such the stand-alone conductor the input-output type description of the form

$$i_o = g_1 v_i + g_2 v_i^2 + g_3 v_i^3 + ... \quad (16)$$

Furthermore, we can rewrite (16) as

$$i_o^{(1)} + i_o^{(2)} + i_o^{(3)} + ... = g_1 v_i + g_2 v_i^2 + g_3 v_i^3 + ... \quad (17a)$$

with $i_o^{(1)}$, $i_o^{(2)}$, $i_o^{(3)}$, and so on, standing for components of the current i_o (that is partial responses) associated exclusively with the linear part, with the second-order nonlinearity, with the third-order nonlinearity, and so on, incorporated in the conductor description. We call them: the term of the first (linear), second, and third order, respectively, and so on. Further, according to (17a), they are given by

$$i_o^{(1)} = g_1 v_i, \;\; i_o^{(2)} = g_2 v_i^2, \;\; i_o^{(3)} = g_3 v_i^3, \;\; ... \; . \quad (17b)$$

With regard to (17a), (17b), and (16), we can make the following remarks:

1. The series of expressions in (17b) can be viewed as a description of an iterative process. We calculate first $i_o^{(1)}$, then $i_o^{(2)}$, and $i_o^{(3)}$, and so on.
2. On the other hand, note however that the terms of higher orders in (17b) are fully independent of lower ones (and vice versa). So they can be, in fact, calculated in any order.
3. The Maclaurin series expansion (16) can be viewed as an analog (here for a fully memoryless element) of the Volterra series [2]. But the Volterra series is a typical input-output description of a nonlinear circuit or system.

Consider now the parallel connection of a nonlinear conductor and a nonlinear capacitor, described by (14a) and (14b), respectively. Then, in this case, the resulting current *i* flowing into the block consisting of these elements is given by

$$i = \left(g_1 + c_1 \frac{d}{dt} \right)(v) + \left(g_2 + c_2 \frac{d}{dt} \right)(v^2) + \left(g_3 + c_3 \frac{d}{dt} \right)(v^3) + ... \quad (18a)$$

where

$$y_1 = g_1 + c_1 \frac{d}{dt},\ y_2 = g_2 + c_2 \frac{d}{dt},\ y_3 = g_3 + c_3 \frac{d}{dt} \tag{18b}$$

define the linear operators applied to the signals: v, v^2, and v^3, respectively. The small letter v in (18a) means the voltage across the block made of conductor and capacitor in parallel. This voltage represents its second terminal variable.

Note that we chose above a more compact form with the use of operators on the right-hand side of (18a). In such the notation, the operators are typically put into the parentheses. By the way, observe also that to be fully consistent with this rule (18a) should be written as

$$\begin{aligned} i &= \left(g_1 + c_1 \frac{d}{dt} \right)(v) + \left(g_2 + c_2 \frac{d}{dt} \right)(\cdot)^2 (v) + \left(g_3 + c_3 \frac{d}{dt} \right)(\cdot)^3 (v) + \ldots = \\ &= (y_1)(v) + (y_2)(\cdot)^2 (v) + (y_3)(\cdot)^3 (v) + \ldots \quad . \end{aligned} \tag{18c}$$

In what follows, we prefer, however, to use rather the form applied in (18a).

Having the expressions (18a), (18b), and (18c), let us now consider the parallel connection of a nonlinear conductor and a nonlinear capacitor as a stand-alone block with the terminal variable $v = v_i$ being the input signal and $i = i_o$ being, accordingly, the output signal. Then, it is easy to show that this block (in what follows, we call it also a circuit element) possesses the following input-output type description

$$i_o = (y_1)(v_i) + (y_2)(v_i^2) + (y_2)(v_i^3) + \ldots \quad . \tag{19}$$

Moreover, the counterparts of (17a) and (17b) in this case have the following forms

$$i_o^{(1)} + i_o^{(2)} + i_o^{(3)} + \ldots = (y_1)(v_i) + (y_2)(v_i^2) + (y_2)(v_i^3) + \ldots \tag{20a}$$

and

$$i_o^{(1)} = (y_1)(v_i),\ i_o^{(2)} = (y_2)(v_i^2),\ i_o^{(3)} = (y_3)(v_i^3),\ \ldots \tag{20b}$$

respectively. Furthermore, with regard to (20a) and (20b), we can make similar remarks as those made before with respect to (17a) and (17b). Also, observe that (19) and the Volterra series have the forms of sums of operators working with the corresponding powers of the input signal v_i (v_i or v_i^2 or v_i^3, and so on). So, in the sense of such the form of representation, (19) can be viewed as an analog of the Volterra series.

It is also important to observe that the left-hand sides of the equalities in (20b) stand for the names of the operators $\left(i_o^{(1)}\right)(v_i)$, $\left(i_o^{(2)}\right)(v_i^2)$, $\left(i_o^{(3)}\right)(v_i^3)$, and so on. That is $i_o^{(1)}$ is short for $\left(i_o^{(1)}\right)(v_i)$, $i_o^{(2)}$ for $\left(i_o^{(2)}\right)(v_i^2)$, $i_o^{(3)}$ for $\left(i_o^{(3)}\right)(v_i^3)$, and so on. We will use this terminological convention also in what follows.

After explaining the forms of the input-output type descriptions for a nonlinear conductor alone and its parallel connection with a nonlinear capacitor - as the stand-alone circuits - consider now another situation when they are a part of a network and the input signal is applied to a place different from their terminals. Assume that the input signal applied to the nonlinear network considered is the voltage. Denote it as v_i. Furthermore, assume that the relationships between the voltages (except v_i) and currents in the network, that is the terminal variables of its elements, and the input signal v_i can be expressed by the Volterra series

$$v(t) = \sum_{n=1}^{\infty} v^{(n)}(t) \text{ and } i(t) = \sum_{n=1}^{\infty} i^{(n)}(t) \ . \tag{21a}$$

In (21a), $v^{(n)}(t)$ and $i^{(n)}(t)$ represent the partial responses in the current and voltage, respectively, associated with the corresponding orders, $n=1$ (linear), $2, 3, \ldots$, of the network nonlinearities. These responses are the corresponding terms in the Volterra series and are given by [2], [6], [7]

$$v^{(n)}(t) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} h_V^{(n)}(\tau_1, \ldots, \tau_n) \prod_{k=1}^{n} v_i(t-\tau_k) d\tau_k \tag{21b}$$

and

$$i^{(n)}(t) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} h_I^{(n)}(\tau_1, \ldots, \tau_n) \prod_{k=1}^{n} v_i(t-\tau_k) d\tau_k \tag{21c}$$

for the input signal $v_i(t)$ applied to the nonlinear network with one (single) input port. In (21b), $h_V^{(n)}(\tau_1, \ldots, \tau_n)$ means the network nonlinear impulse response of the n-th order regarding the voltage across the element considered and related to the network input port assumed.

Similarly, $h_I^{(n)}(\tau_1,...,\tau_n)$ in (21c) is the network nonlinear impulse response of the n-th order regarding the current flowing through the above element and related to the aforementioned input port.

Dropping the argument *t* in (21a) as well as on the left-hand sides of (21b) and (21c), for simplicity of notation, as was also done in the previous section, and introducing then the expressions (21a) in (14a), we obtain

$$\begin{aligned} i = i^{(1)} + i^{(2)} + i^{(3)} + ... = g_1 v^{(1)} + \left(g_1 v^{(2)} + g_2 v^{(1)} v^{(1)} \right) + \\ + \left(g_1 v^{(3)} + g_2 v^{(1)} v^{(2)} + g_2 v^{(2)} v^{(1)} + g_3 v^{(1)} v^{(1)} v^{(1)} \right) + ... \end{aligned} \tag{22}$$

for a nonlinear conductor.

Note that the components occurring on both sides of (22) can be interpreted in the terminology of operators. That is $i^{(1)} = \left(i^{(1)}\right)\left(v_i\right)$, $i^{(2)} = \left(i^{(2)}\right)\left(v_i^2\right), \dots\dots, g_1 v^{(1)} = \left(g_1 v^{(1)}\right)\left(v_i\right), \left(g_1 v^{(2)} + g_2 v^{(1)} v^{(1)}\right) = \left(g_1 v^{(2)} + g_2 v^{(1)} v^{(1)}\right)\left(v_i^2\right)$, and so on, are the operators (whose names are put into the first parentheses) working with the corresponding powers of the input signal v_i (v_i or v_i^2 or v_i^3, and so on). Moreover, note that $g_1 v^{(2)} = \left(g_1\right)\left(v^{(2)}\right)$ is a composite operator made of g_1 and $v^{(2)}$, $\left(g_1 v^{(2)} + g_2 v^{(1)} v^{(1)}\right) = \left(g_1\right)\left(v^{(2)}\right) + \left(g_2\right)\left(\cdot\right)^2\left(v^{(1)}\right)$ is a sum of two composite operators indicated, etc.

Comparison of the components of the same order (degree) with respect to the input signal v_i on both sides of (22) gives

$$i^{(1)} = g_1 v^{(1)} \tag{23a}$$

$$i^{(2)} = g_1 v^{(2)} + g_2 v^{(1)} v^{(1)} \tag{23b}$$

$$i^{(3)} = g_1 v^{(3)} + g_2 v^{(1)} v^{(2)} + g_2 v^{(2)} v^{(1)} + g_3 v^{(1)} v^{(1)} v^{(1)} \tag{23c}$$

and so on. The correctness of the procedure used in (23a-c) is explained in the appendix.

Proceeding similarly for our second illustrative example, which is the parallel connection of nonlinear conductor and capacitor, and using (18c) and (21a), we get

$$i = i^{(1)} + i^{(2)} + i^{(3)} + \ldots =$$
$$= (y_1)(v^{(1)}) + ((y_1)(v^{(2)}) + (y_2)(v^{(1)}v^{(1)})) +$$
$$+ ((y_1)(v^{(3)}) + (y_2)(v^{(1)}v^{(2)}) + (y_2)(v^{(2)}v^{(1)}) + (y_3)(v^{(1)}v^{(1)}v^{(1)})) + \ldots \quad . \tag{24}$$

Then, equating to each other, as before, the expressions of same order (degree) on both sides of (24), we obtain successively

$$i^{(1)} = (y_1)(v^{(1)}) \tag{25a}$$

$$i^{(2)} = (y_1)(v^{(2)}) + (y_2)(v^{(1)}v^{(1)}) \tag{25b}$$

$$i^{(3)} = (y_1)(v^{(3)}) + (y_2)(v^{(1)}v^{(2)}) + (y_2)(v^{(2)}v^{(1)}) + (y_3)(v^{(1)}v^{(1)}v^{(1)}) \tag{25c}$$

and so on.

Now, with regard to (22), (23a-c), (24), and (25a-c), we can make the following remarks:

1. The series of expressions given by (23a-c) and (25a-c) represent the descriptions of the iterative processes. That is we calculate first $i^{(1)}$ and $v^{(1)}$, then $i^{(2)}$ and $v^{(2)}$, and $i^{(3)}$ and $v^{(3)}$, and so on.
2. The order of calculations of the aforementioned terms is however now fixed. That is, for example, the calculation of the term $i^{(2)}$ can not precede that of the term $i^{(1)}$ because, according to (23b) or (25b), to calculate $i^{(2)}$ we need to know $v^{(1)}$. Hence, the calculation of $v^{(1)}$ must be carried out in one of the preceding steps. And this is really so done here, in the first step represented by (23a) or (25a). Furthermore, note that the same arguments regard also the next steps.
3. The series (22) and (24) are generally (in the case of networks possessing memory) the Volterra series. At first glance, this is not, maybe, easy to observe. Remember, however, that the successive terms $i^{(1)}$, $v^{(1)}$, $i^{(2)}$, $v^{(2)}$, $i^{(3)}$ and $v^{(3)}$, and so on, in (22) and (24) are the partial responses of the corresponding Volterra series working with the input signal v_i applied to the network. Moreover, the operators (g_1), (y_1), (g_2), (y_2), (g_3), and (y_3), and so on, are linear operators. So their resulting effect in (22) and (24) is of transforming only one Volterra series into another. Therefore, the series (22) and (24) can be also viewed as the input-output relationships for the whole network considered with the input being at the place where the input signal v_i is applied (to the network) and the output at the terminal where the

terminal variable current *i* occurs. Obviously, it is not any input-output relationship for the element embedded.

Concluding, from the remarks presented beneath (17b) and (25a-c), it is evident what are the common and different features of the two types of descriptions discussed in this section.

4 Input-output and in-network type descriptions in multi-frequency domains

Usually weakly nonlinear circuits described by the Volterra series are analyzed in such a way that the descriptions formulated in the time domain are transformed into the frequency domain (strictly saying, into the multi-frequency domains) in the first step. This is done for a purpose of evaluating the so-called nonlinear transfer functions [2], [7] of a given circuit. In the second step, the nonlinear transfer functions found are used to evaluate the response of a given circuit to a single sinusoidal or a sum of sinusoidal signals of different frequencies. For more details, see for example [2], [7]. So, in such the analysis, we need to have the descriptions of basic nonlinear circuit elements in the frequency domain. We illustrate this point briefly now by performing the aforementioned transformations of the input-output and in-network type descriptions of the parallel connection of nonlinear conductor and capacitor derived in the previous section.

To transform the expressions given by (20b), (25a), (25b), and (25c) into the multi-frequency domains, one needs first to introduce in them artificial auxiliary time variables. And this is done by a standard procedure as described, for instance, in [2], [7]. That is $i_o^{(n)} = i_o^{(n)}(t) \to i_o^{(n)}(t_1,..,t_n)$ is applied for currents on the left-hand sides of equalities in (20b) and $v_i^n = v_i^{(n)}(t) \to v_i^n(t_1,..,t_n)$ with

$$v_i^{(n)}(t_1,..,t_n) = v_i(t_1)\cdots v_i(t_n) \tag{26a}$$

for powers of v_i which occur on the right-hand sides of equalities in (20b). Further, one writes for $i^{(n)} = i^{(n)}(t) \to i^{(n)}(t_1,..,t_2)$ with

$$i^{(n)}(t_1,..,t_n) = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} h_I^{(n)}(\tau_1,..,\tau_n) \prod_{k=1}^{n} v_i(t_k - \tau_k) d\tau_k \tag{26b}$$

and finally, for $v^{(n)} = v^{(n)}(t) \to v^{(n)}(t_1,..,t_2)$ with

$$v^{(n)}(t_1,..,t_n)=\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty}h_V^{(n)}(\tau_1,..,\tau_n)\prod_{k=1}^{n}v_i(t_k-\tau_k)d\tau_k \tag{26c}$$

for (25a-c). In (26a-c), $t_1,..,t_n$ mean the artificial auxiliary time variables, and $n=1,2,3,...$.

Afterwards, we use the so-called multidimensional Fourier transforms [2], [7] to (20b), (25a), (25b), and (25c), in which the quantities mentioned above are written out with the corresponding numbers of artificial time variables $t_1,..,t_n$. These Fourier transforms, for the successive indices $n=1,2,3,...$, are defined by

$$G^{(n)}(f_1,...,f_n)=\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty}g^{(n)}(t_1,...,t_n)\exp(-j2\pi(f_1t_1+...+f_nt_n))dt_1\cdots dt_n \tag{27}$$

where $G^{(n)}(f_1,...,f_n)$ means the n-dimensional Fourier transform of a function $g^{(n)}(t_1,...,t_n)$ having *n* arguments being artificial auxiliary time variables. Moreover, $f_1,...,f_n$ in (27) are the frequencies from the n-dimensional frequency space.

As a result, we get for (20b)

$$I_o^{(1)}(f_1)=Y_1(f_1)V_i(f_1) \tag{28a}$$

$$I_o^{(2)}(f_1,f_2)=Y_2(f_1+f_2)V_i(f_1)V_i(f_2) \tag{28b}$$

$$I_o^{(3)}(f_1,f_2,f_3)=Y_3(f_1+f_2+f_3)V_i(f_1)V_i(f_2)V_i(f_3) \tag{28c}$$

and so on. In (28a), (28b), and (28c), f_1, f_2, and f_3 represent the frequency variables in the corresponding multidimensional frequency spaces. That is f_1 belongs to the one-dimensional frequency space, f_1 and f_2 belong to the two-dimensional frequency space, f_1, f_2, and f_3 belong to the three-dimensional frequency space, and so on. V_i means the one-dimensional Fourier transform of v_i. Furthermore, from the form of the operators y_1, y_2, and y_3, which are defined by (18b), it follows [2], [7] that their one-, two-, and three-dimensional, respectively, Fourier transforms occurring in (28a-c) are given by

$$Y_1\left(f_1\right)=g_1+j2\pi f_1c_1 \tag{29a}$$

$$Y_2\left(f_1+f_2\right)=g_2+j2\pi\left(f_1+f_2\right)c_2 \tag{29b}$$

$$Y_3\left(f_1+f_2+f_3\right)=g_3+j2\pi\left(f_1+f_2+f_3\right)c_3\ . \tag{29c}$$

Note now that among the quantities Y_1, Y_2, and Y_3 in (29a-c) only one Y_1 is the admittance which we know from the theory of linear circuits. Only this quantity is expressed in the unit siemens=amp/volt (S=A/V). The unit of Y_2 is A/(V)^2, and of Y_3 A/(V)^3.

In [1], the notion of the nonlinear admittance has been introduced as a nonlinear circuit element characterized by the following set of *coefficients* $\{Y_1,Y_2,Y_3\}$. Another name was also used for this set, namely in [5], its elements were called the (first) three Taylor coefficients of the (nonlinear) admittance *Y*.

Note now that the quantities Y_1, Y_2, and Y_3 in the input-output type description (in the multi-frequency domains) given by (28a-c) play a role of the circuit nonlinear transfer functions [2], [7] (of the voltage to current type) because from (28a-c) we can get

$$Y_1\left(f_1\right)=I_o^{(1)}\left(f_1\right)\big/V_i\left(f_1\right) \tag{30a}$$

$$Y_2\left(f_1+f_2\right)=I_o^{(2)}\left(f_1,f_2\right)\big/\left(V_i\left(f_1\right)V_i\left(f_2\right)\right) \tag{30b}$$

$$Y_3\left(f_1+f_2+f_3\right)=I_o^{(3)}\left(f_1,f_2,f_3\right)\big/\left(V_i\left(f_1\right)V_i\left(f_2\right)V_i\left(f_3\right)\right) \tag{30c}$$

and so on. (For more basics on nonlinear transfer functions, see for example [2] and [7].)

Furthermore, applying (26b), (26c), and (27) to (25a-c) gives

$$I^{(1)}(f_1)=Y_1\left(f_1\right)V^{(1)}(f_1) \tag{31a}$$

$$I^{(2)}(f_1,f_2)=Y_1\left(f_1+f_2\right)V^{(2)}(f_1,f_2)+Y_2\left(f_1+f_2\right)V^{(1)}(f_1)V^{(1)}(f_2) \tag{31b}$$

$$I^{(3)}(f_1,f_2,f_3)=Y_1\left(f_1+f_2+f_3\right)V^{(3)}(f_1,f_2,f_3)+Y_2\left(f_1+f_2+f_3\right)\cdot$$
$$\cdot\left(V^{(1)}(f_1)V^{(2)}(f_2,f_3)+V^{(2)}(f_1,f_2)V^{(1)}(f_3)\right)+Y_3\left(f_1+f_2+f_3\right)V^{(1)}(f_1)V^{(1)}(f_2)V^{(1)}(f_3) \tag{31c}$$

and so on.

Note that (31a), (31b), and (31c) can be further rewritten in the following form

$$I^{(1)}(f_1) = Y_1\left(f_1\right)V^{(1)}(f_1) + S_I^{(1)}(f_1) \tag{32a}$$

$$I^{(2)}(f_1, f_2) = Y_1\left(f_1 + f_2\right)V^{(2)}(f_1, f_2) + S_I^{(2)}(f_1, f_2) \tag{32b}$$

$$I^{(3)}(f_1, f_2, f_3) = Y_1\left(f_1 + f_2 + f_3\right)V^{(3)}(f_1, f_2, f_3) + S_I^{(3)}(f_1, f_2, f_3) \tag{32c}$$

with

$$S_I^{(1)}(f_1) = 0 \tag{33a}$$

$$S_I^{(2)}(f_1, f_2) = Y_2\left(f_1 + f_2\right)V^{(1)}(f_1)V^{(1)}(f_2) \tag{33b}$$

$$\begin{aligned} S_I^{(3)}(f_1, f_2, f_3) = {} & Y_2\left(f_1 + f_2 + f_3\right)\Big(V^{(1)}(f_1)V^{(2)}(f_2, f_3) + \\ & + V^{(2)}(f_1, f_2)V^{(1)}(f_3)\Big) + Y_3\left(f_1 + f_2 + f_3\right)V^{(1)}(f_1)V^{(1)}(f_2)V^{(1)}(f_3)\ . \end{aligned} \tag{33c}$$

and so on. We interpret the quantities $S_I^{(1)}(f_1)$, $S_I^{(2)}(f_1, f_2)$, $S_I^{(3)}(f_1, f_2, f_3)$ in (33a-c), and the corresponding ones of higher orders, as the independent current sources. That is $S_I^{(1)}$ independent of $V^{(n)}$, $n \geq 1$, $S_I^{(2)}$ independent of $V^{(n)}$, $n \geq 2$, $S_I^{(3)}$ independent of $V^{(n)}$, $n \geq 3$, and so on). According to (32a-c), these independent current sources are connected in parallel with Y_1 (that is with the admittance coming from the linear part of the description of the connection of nonlinear conductor and capacitor considered).

Equations (31a-c) or, equivalently, (32a-c) with (33a-c) constitute the in-network type description in the frequency domains. Comparison of its form with that of the input-output type shows that it is evidently different from the latter given by (28a-c). It is named in this paper the description of in-network type because of its use as a stamp [8] in construction of the modified admittance matrix of the (whole) network. Also, it is worthy to note that the idea of writing the in-network type description in the form given by (32a-c), with the use of the independent sources, appeared for the first time in the literature in [9]. It was also exploited in [2], a mile-stone paper in the area.

Now, we present briefly how the in-network type descriptions as stamps are used in construction of the network matrix occurring in the so-called modified nodal formulation (MNF) [6], [8]. To this end, we start first with the description of linear circuits in the frequency domain. We remind that the basic circuit elements can be split into two groups: one formed by elements possessing an admittance description, and the second consisting of those that do not. Further, note that the elements which do not possess the admittance description can be included in such a way that they extend the classic nodal formulation of circuit equations (based on the use of the admittance matrix) by new matrices $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{Z}$. Then, we arrive at the so-called modified nodal formulation (the modified admittance matrix description), which has the following form [8]

$$\begin{bmatrix} \boldsymbol{Y} & \boldsymbol{\alpha} \\ \boldsymbol{\beta} & \boldsymbol{Z} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_N \\ \boldsymbol{I}_B \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_I \\ \boldsymbol{X}_V \end{bmatrix} \tag{34}$$

where $\boldsymbol{Y}$ is an admittance matrix determined by the circuit elements possessing the admittance description, but $\boldsymbol{Z}$ is a matrix of the impedance character determined by the non-admittance-type elements. Furthermore, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are dimensionless matrices, $\left[\boldsymbol{V}_N \ \ \boldsymbol{I}_B\right]^T$ is a vector of unknown nodal voltages and branch currents, but $\left[\boldsymbol{X}_I \ \ \boldsymbol{X}_V\right]^T$ is a vector of the external independent current $(\boldsymbol{X}_I)$ and voltage $(\boldsymbol{X}_V)$ sources applied to the circuit inputs. The letter T in the above vectors stands for the transposing operation. Moreover, the elements of the matrices and vectors occurring in (34) depend, generally, upon the frequency. Finally, we mention also that the modified admittance matrix occurring in (34) is most often formed (specifically for computer programs) with the use of a mnemonic technique of so-called element stamps [8]. And such the element stamps are nothing else than the constitutive equations of the linear elements in the frequency domain.

Knowing the form of the descriptions of basic nonlinear elements in the multi-frequency domains, which is illustrated here by equations (32a-c) and (33a-c) of our example, it is easy to deduce that the MNF for weakly nonlinear circuits having a single input port is a scheme of an iterative process as expressed by

$$\begin{bmatrix} \boldsymbol{S}_I^{(n)} \\ \boldsymbol{S}_V^{(n)} \end{bmatrix} + \begin{bmatrix} \boldsymbol{Y}^{(n)} & \boldsymbol{\alpha}^{(n)} \\ \boldsymbol{\beta}^{(n)} & \boldsymbol{Z}^{(n)} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_N^{(n)} \\ \boldsymbol{I}_B^{(n)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_I^{(n)} \\ \boldsymbol{X}_V^{(n)} \end{bmatrix} . \tag{35}$$

In (35), the superscript $n=1$ (linear), 2, 3, …means the order of the analysis. Moreover, $\boldsymbol{S}_{\boldsymbol{I}}^{(\boldsymbol{n})}$ and $\boldsymbol{S}_{\boldsymbol{V}}^{(\boldsymbol{n})}$ are the vectors of the internal independent current and voltage sources, respectively, coming from the descriptions of the circuit nonlinear elements. Furthermore, note that the vector $\left[\boldsymbol{X}_{\boldsymbol{I}}^{(\boldsymbol{n})}\ \boldsymbol{X}_{\boldsymbol{V}}^{(\boldsymbol{n})}\right]^{T}$ for the external independent sources, occurring on the right-hand side of (35), has the form

$$\begin{bmatrix} \boldsymbol{X}_{\boldsymbol{I}}^{(1)} \\ \boldsymbol{X}_{\boldsymbol{V}}^{(1)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_{\boldsymbol{I}} \\ \boldsymbol{X}_{V} \end{bmatrix} \text{ and } \begin{bmatrix} \boldsymbol{X}_{\boldsymbol{I}}^{(\boldsymbol{n})} \\ \boldsymbol{X}_{\boldsymbol{V}}^{(\boldsymbol{n})} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \text{ for } n>1. \tag{36}$$

For more details, see [6].

Consider now an example of a weakly nonlinear network in which the input signal $X_{Vs}=V_i$ (that is the s-th element of the vector $\boldsymbol{X}_V$ is equal to V_i) is applied to a node, say, *i* (for simplicity of notation) and the output of this network is at a node numbered *m*. Further, solving (35) successively for $n=1,2,3,...$, we get a series of vectors of nodal voltages $\boldsymbol{V}_{\boldsymbol{N}}^{(1)}$, $\boldsymbol{V}_{\boldsymbol{N}}^{(2)}$, $\boldsymbol{V}_{\boldsymbol{N}}^{(3)}$, and so on. The m-th elements in these vectors will be the voltages $V_m^{(1)}$, $V_m^{(2)}$, $V_m^{(3)}$, and so on. Now, remember that these voltages are the multidimensional Fourier transforms of partial responses of a Volterra series, so have the form given by (21b). Therefore, according to (21b), (26c), and (27), they can be expressed as

$$V_m^{(n)}\left(f_1,...,f_n\right)=H_{Vm}^{(n)}\left(f_1,...,f_n\right)V_i\left(f_1\right)\cdots V_i\left(f_n\right) \tag{37a}$$

where $H_{Vm}^{(n)}\left(f_1,..,f_n\right)$ means the nonlinear transfer function of the n-th order from the node *i* to the node *m*. So for a special choice of the input signal $V_i=1$, we get from (37a)

$$H_{Vm}^{(n)}\left(f_1,...,f_n\right)=V_m^{(n)}\left(1;f_1,...,f_n\right) \tag{37b}$$

where $V_m^{(n)}\left(1;f_1,...,f_n\right)$ means $V_m^{(n)}\left(f_1,...,f_n\right)$ when $V_i=1$.

Equation (37b) shows the principle of an algorithm for calculating the nonlinear transfer functions of a weakly nonlinear network with a single input port. Then, using the MNF framework, these transfer functions are simply identified with the corresponding nodal voltages and/or branch currents in the analyses of the corresponding orders under the network excitation by a signal equal to one (in the frequency domain). Moreover, note that such the algorithm is very suitable for computer calculations.

In summary, observe first that if we agree to use the notion of the nonlinear admittance [1] (defined as the set of coefficients $\{Y_1, Y_2, Y_3\}$), we must keep in mind the fact that its usage in the in-network type description is different from that in the input-output one (for example, compare (31a-c) with (28a-c)). And just this difference was the reason in this paper of defining deliberately the two aforementioned types of descriptions. By the way note that in the case of a linear admittance, that is a set $\{Y_1\}$, such the need does not exist (compare (31a) with (28a), for example).

Second, note that equations (31a-c) of our example for the in-network type description can be also interpreted otherwise, as was shown in (32a-c) and (33a-c). With regard to the latter, we say that in each iteration the parallel connection of the nonlinear conductor and capacitor is represented by a parallel connection of their linear parts and the corresponding independent (for a given iteration) current source, in parallel as well. The values of these elements are calculated for the prescribed sets of frequencies. Such the procedure results in a single modified admittance matrix (evaluated in the case of a network with memory for different frequencies, depending upon the iteration number) for the whole network, as shown in (35). Then, all the coefficients Y_2 and Y_3 are hidden in the source vectors $\boldsymbol{S}_{\boldsymbol{I}}^{(2)}$, $\boldsymbol{S}_{\boldsymbol{V}}^{(2)}$, $\boldsymbol{S}_{\boldsymbol{I}}^{(3)}$, and $\boldsymbol{S}_{\boldsymbol{V}}^{(3)}$ occurring on the left-hand side of (35).

5 Remark on a certain operator introduced in [1]

In [1], an operator called "$\circ$" was introduced. However, its definition given therein is highly imprecise. So its usage can lead to misleading results. Because of lack of space, we point here only out, using the representation of our example given by (18c), that the correct use in the time domain of the operator named "$\circ$"should look like as shown below

$$\begin{aligned} i(t) &= \Big((y_1)+(y_2)(\cdot)^2+(y_3)(\cdot)^3+\dots\Big)(v)(t) = \Big((y_1)+(y_2)(\cdot)^2+\dots\Big)\Big(v^{(1)}+v^{(2)}+\dots\Big)(t) = \\ &= (y_1)\Big(v^{(1)}+v^{(2)}+v^{(3)}+\dots\Big)(t)+(y_2)\Big(v^{(1)}+v^{(2)}+v^{(3)}+\dots\Big)^2(t)+ \\ &\quad +(y_3)\Big(v^{(1)}+v^{(2)}+v^{(3)}+\dots\Big)^3(t)+\dots \end{aligned} \tag{38}$$

where the argument *t* of the current and voltage, and of the partial voltage responses is also shown.

Moreover, observe that the direct usage in the frequency domain, as can be misleadingly understood reading [5], of the operator "$\circ$" is not possible. This follows from the fact that, as in our example, $V^{(1)}(f_1)$, $V^{(2)}(f_1, f_2)$, and $V^{(3)}(f_1, f_2, f_3)$ can not be added to each other because they belong to different frequency spaces.

Appendix

The correctness of the procedure used in (23a-c) relies upon the fact that according to (26b) and (26c), and the terminology used in [10] the operators $i^{(1)}$ and $g_1 v^{(1)}$ are the first order polynomial mappings of v_i, $i^{(2)}$ and $\left(g_1 v^{(2)} + g_2 v^{(1)} v^{(1)}\right)$ are the second order polynomial mappings of v_i, $i^{(3)}$ and $\left(g_1 v^{(3)} + g_2 v^{(1)} v^{(2)} + g_2 v^{(2)} v^{(1)} + g_3 v^{(1)} v^{(1)} v^{(1)}\right)$ are the third order polynomial mappings of v_i, and so on. Denote them, respectively, as $M^{(1)}(v_i)$ and $P^{(1)}(v_i)$, $M^{(2)}(v_i)$ and $P^{(2)}(v_i)$, $M^{(3)}(v_i)$ and $P^{(3)}(v_i)$, and so on. Further, observe that such the polynomial mappings are homogeneous of the corresponding degrees *n,* $n = 1, 2, 3, \ldots$, respectively. More precisely, this means that $M^{(1)}(\alpha v_i) = \alpha M^{(1)}(v_i)$, $M^{(2)}(\alpha v_i) = \alpha^2 M^{(1)}(v_i)$, $M^{(3)}(\alpha v_i) = \alpha^3 M^{(3)}(v_i)$, and so on, where α is a real number. Now, see that we can postulate fulfillment of $M^{(1)}(v_i) = P^{(1)}(v_i)$ for all the possible values of the input variable v_i. This does not contradict the homogeneity property of these operators because really $M^{(1)}(\alpha v_{i1}) = P^{(1)}(\alpha v_{i1})$ holds for all the values of α and any chosen $v_i = v_{i1}$. Similarly, the postulated fulfillment of $M^{(2)}(v_i) = P^{(2)}(v_i)$ for all the possible values of v_i does not contradict the homogeneity property of these mappings because $M^{(2)}(\alpha v_{i1}) = P^{(2)}(\alpha v_{i1})$ holds as well for all the values of α and any chosen $v_i = v_{i1}$. And this reasoning can be continued for higher orders of the polynomial mappings defined above. However, postulating, for example, fulfillment of $M^{(2)}(v_i) = P^{(3)}(v_i)$ for all the possible values of the input variable v_i contradicts the homogeneity property of these mappings. To see this, observe that, for a chosen $v_i = v_{i1}$, there are such values of α for which

$M^{(2)}(\alpha v_{i1}) = \alpha^2 M^{(2)}(v_{i1}) \neq \alpha^3 P^{(3)}(v_{i1}) = P^{(3)}(\alpha v_{i1})$ holds. The latter, however, contradicts the assumption that $M^{(2)}(v_i) = P^{(3)}(v_i)$ for any v_i. And this ends the proof of correctness of the procedure used in (23a-c).

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WYKORZYSTANIE FORMALIZMU ADMITANCYJNEGO W ANALIZIE NIELINIOWEJ

Streszczenie – W pracy zostały omówione podstawowe zagadnienia związane z użyciem formalizmu admitancyjnego w analizie w dziedzinie częstotliwości układów elektronicznych z tzw. małymi nieliniowościami. Pokazano, że konieczne jest w niej rozróżnienie pomiędzy modelami typu wejście-wyjście a tzw. modelami wewnętrznymi układów lub ich elementów składowych. Pozwala to na uniknięcie błędów w opisie układów. Pokazano również, jaki jest związek pomiędzy pojęciem nieliniowej admitancji, wprowadzonym w jednym z ostatnio opublikowanych artykułów, a zmodyfikowaną metodą napięć węzłowych dla układów z małymi nieliniowościami (w której to wykorzystuje się tzw. zmodyfikowaną macierz admitancyjną).