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WEIGHT DISTRIBUTION IN SOME LINEAR CODES

Summary - Paper describes the qualitative properties of linear codes in the small dimension and codimension, especially the same like oryginal Hamming code. The codes are constructed by algorithms based on „Monte Carlo" method. We considered four dimensions codes embedded in seven dimensional space over the field with twenty five elements. An analysis of the Hamming distance for such a codes is also represented in the paper.

Keywords: Linear codes, Grassman mainifolds, Hamming distance

1. Introduction

Linear codes are commonly used in data transmission in noisy transmission medium. ([1]) By the dimensions of code is meant bandwidth for the transmitted information, the co-dimension code, however, measures the so-called redundancy that is, the amount of information necessary for the detection and possible error correction of transmitted data.

Linearity simplifies code encoding and decoding, which results in high efficiency of implemented algorithms. For obvious reasons, the binary codes are most commonly used, which greatly narrows the range of possible quality of the code. Using a larger number of states (of finite characteristics of more than two), it provides a flexible structure collection of linear codes. ([2]) Effective and quick method for generating such a codes was pressented in work [3].

2. Hamming metrics

Definition. *Expanding the discrete metrics (the only one) in the field* $\mathbb{F}_q$

$$\|x\| = \begin{cases} 0 & dla \quad x = 0 \\ 1 & dla \quad x \neq 0 \end{cases}$$

to the norm L^1 *in vector space* $\mathbb{F}_q^n$ *with*

$$\|v\| = \sum_{j=1}^{n} \|v_j\|, \; v = (v_1, \ldots, v_n).$$

we get, so called Hamminga metrics

$$\varrho\colon \mathbb{F}_q^n \times \mathbb{F}_q^n \longrightarrow \mathbb{Z}_+, \quad \varrho(u,v) = \|u-v\|.$$

Definition. *For linnear code* $C \subset \mathbb{F}_q^n$

$$d = \min_{u,v\in C, u\neq v} \varrho(u,v) = \min_{u,v\in C, u\neq v} \|u-v\| = \min_{v\in C, v\neq 0} \|v\|.$$

we call the minimal Hamming distance *or just the* Hamming distance *([4]).*

Hamming distance for a code $C \subset \mathbb{F}_q^n$ is equal to the minimal number of linearly dependent columns of code anihilator C.

We can detect d code errors C and we can correct $\llcorner \frac{d-1}{2} \lrcorner$ of them ([4]).

3. Schubert decomposition

Folklore. $\mathbb{F}$ *any field,* $k, n \in \mathbb{N}$ *arbitrary natular numbers, such that* $k \leqslant n$*. Let* B *be any matrix with* n *rows and* k *columns over field* $\mathbb{F}$ *and with maximal rank* (k)*. This matrix can represent* k*-dimensional vector subspace of the* n*-dimensional vector space* $\mathbb{F}^n$

$$B = \begin{bmatrix} b_{1,1} & \dots & b_{1,k} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \dots & b_{n,k} \end{bmatrix}.$$

If we applay standard Gauß elimination method on the matrix B *in right-left and bottom-top direction (acting on columns only), we get*

$$B \cdot g = \begin{bmatrix} * & * & \dots & * & * \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & \dots & * & * \\ 1 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & * & * \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & * & * \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & * \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & * \\ 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow \sigma_1 \\ \\ \\ \\ \\ \\ \\ \\ \leftarrow \sigma_k \\ \\ \\ \\ \end{matrix}$$

(g *is square* k *by* k *invertible matrix - from Gauß elimination algorithm) so called* echelon form *of* B*, where stars* $*$ *are arbitrary elements of the field* $\mathbb{F}$*. Let we enumerate rows with the single unit entry, by numbers* $\sigma_1, \dots, \sigma_k$*, where*

$1 \leqslant \sigma_1 < \ldots < \sigma_k \leqslant n$. *Sequence* $\sigma = (\sigma_1, \ldots, \sigma_k)$ *we call* Schubert symbol. *The above matrix represents set of all* k*-dimensional vector subspaces* W *of vector space* $\mathbb{F}^n$ *satisfying following condition:*

$$\dim_{\mathbb{F}} (W \cap Fl^{\sigma_j}) = j \ \wedge \ \dim_{\mathbb{F}} (W \cap Fl^{\sigma_j - 1}) = j - 1, \ \ for \ \ all \ \ j = 1, 2, \ldots, k,$$

where by Fl^j *we mean* j*-flag element, i.e.* j*-dimensional vector subspace spanned by first* j *vectors from standard basis of the linear space* $\mathbb{F}^n$.

Let $e(\sigma)$ *a set of such a subspaces of* W, *represented by the matrix* $B \cdot g$ *or we can rather say, by the* Schubert symbol

$$\sigma = (\sigma_1, \ldots, \sigma_k), \ \ 1 \leqslant \sigma_1 < \ldots < \sigma_k \leqslant n)$$

We can state, that set $e(\sigma)$ *has a structure of so called* Schubert cell*, and if we count the stars* $*$ *we find its dimension.*

$$\dim_{\mathbb{F}} \ e(\sigma) = \sum_{j=1}^{k} \ (\sigma_j - j).$$

Let $p(r)$ *be a number or cells of dimension* r. *Using well known asymptotic expansion done by Ramanuja we get:*

$$p(r) \approx \sqrt{3} \cdot \frac{e^{\pi\sqrt{\frac{2r}{3}}}}{4r} \quad [5]$$

Denote by $[n] = \{1, \ldots, n\}$. Let $[n, k]$ be a set of all k-element subsets of $[n]$. Let

$$order : [n, k] \longrightarrow \binom{[n]}{k}$$

be a natural enumeration of elements, i.e. for any $S \in [n, k]$ denote $\sigma = order(S)$, $1 \leq \sigma_1 < \ldots < \sigma_k \leq n$ and $S = \{\sigma_1, \ldots, \sigma_k\}$.

Now we can reorganize enumeration of axis. Let a be any permutation of $[n]$. For any $S \in [n, k]$ let $a(S)$ be an image of S and we can apply function $order$ to it and detoted it by τ, i.e. $\tau = order(a(S))$. Now let $\sigma = order(S)$ so we get $order \circ a(?) \circ order^{-1}$ mapping of $\binom{[n]}{k}$ to itself. So we get the left action of symmetric group S_n on the set $\binom{[n]}{k}$, it means $\tau = a \cdot \sigma$, $1 \leq \sigma_1 < \ldots < \sigma_k \leq n$, $1 \leq \tau_1 < \ldots < \tau_k \leq n$,

$$\{\tau_1, \ldots, \tau_k\} = \{a_{\sigma_1}, \ldots, a_{\sigma_k}\}$$

$$(ab) \cdot \sigma = a \cdot (b \cdot \sigma) \quad for \ \ a, b \in S_n, \ \ \sigma \in \binom{[n]}{k}$$

For any permutation $a \in S_n$ let us consider the following flag of subspaces

$$Fl_a^1 \subset Fl_a^2 \subset \ldots \subset Fl_a^n$$

where

$$Fl_a^j = \mathrm{span}\{u_{a_1}, \ldots, u_{a_j}\}$$

for $(u_1, \ldots, u_n)$ as a standard basis of $\mathbb{F}^n$.

Let for any $a \in S_n$ and $\sigma \in \binom{[n]}{k}$ define the set of all k-dimensional subspaces W of vector space $\mathbb{F}^n$ by the collection of following conditions

$$\dim_{\mathbb{F}} (W \cap Fl_a^{(a\cdot\sigma)_j}) = j \ \wedge \ \dim_{\mathbb{F}} (W \cap Fl_a^{(a\cdot\sigma)_j - 1}) = j - 1$$

for all $j = 1, 2, \ldots, k$. Above set, denoted by

$$e(a, \sigma) \subset Graß(k, n, \mathbb{F})$$

actually forms a cell of dimension

$$\dim_{\mathbb{F}} e(a, \sigma) = \sum_{j=1}^{k} ((a \cdot \sigma)_j - j).$$

This notation corresponds with this at the beginning of our paper, i.e. $e(\sigma) = e(id, \sigma)$, where id denotes identity permutation in S_n .

We can repeat this action for new order of axis by applying Gauss algorithm to get new echelon form of our subspace. So we get a mapping:

$$ech : Graß(k, n, \mathbb{F}) \times S_n \longrightarrow \binom{[n]}{k}$$

Assume $\tau = ech(W, a)$ where $1 \leq \tau_1 < \ldots < \tau_k \leq n$ and $\{a(\sigma_1), \ldots, a(\sigma_k)\} = \{\tau_1, \ldots, \tau_k\}$.

defining new axis system by renumbering original ones. Cells $e(\sigma)$, where $\sigma_1 = n - k + 1, \ldots, \sigma_k = n$ we will call $fat(a)$ cells. Then

$$fat(a) = e(a_{n-k+1}^{-1}, \ldots, a_n^{-1})$$

$$dim(fat(a)) = k(n - k)$$

We are looking for intersection offat cells for all possible a:

$$\bigcap_{a \in S_n} fat(a)$$

Of course $\bigcap_{a \in S_a} fat(a) \subsetneq fat(\sigma)$ for some σ. and

$$\bigcup_{a \in S_a} fat(a) = Graß(b, n)$$

.

if best codes e.g. codes of Hamming distance of $(n - k + 1)$ exists they must be in that set. The question is if there are such a codes?

We can observe that the intersection could be taken over the some smaller subset of S_n i.e. shuffle permutation (Shf_n).

Theorem.

$$\bigcap_{a \in Shf_n} fat(a) \;=\; \{W \in Gra\beta(k,n,\mathbb{F}); d(W) = n-k+1\}$$

Proof. Let $W \in Gra\beta(k,n,\mathbb{F}); d(W) < n-k+1$ So choose the vector $w \in W$ which achives the minimal distance $|w| = d(W)$, it means that at least k coordinates of w is equal to zero. Now we can shuffle this coordinates to the end (we perform permutation a of the axies). Now we complete our vector to the basis of subspace W. Appling Gauss' algorithm transforming that basis to the echelon form we can observe that this subspace is not a member of $fat(a)$. So we proved that

$$\bigcap_{a \in Shf_n} fat(a) \;\subset\; \{W \in Gra\beta(k,n,\mathbb{F}); d(W) = n-k+1\}$$

To proof the opposite inclusion assume that there exists a shuffle premutation $a \in Shf_n$ of axis such that $W \notin fat(a)$. It means that in this order of axis σ_1 of Schubert symbol is strictly less then $n-k+1$ so $d(W) < n-k+1$. □

Remark. *For some cases there is no space with maximal Hamming distance:*

$$\{W \in Gra\beta(k,n,\mathbb{F}); d(W) = n-k+1\} = \emptyset$$

Theorem.

$$\{W \in Gra\beta(k,n,\mathbb{F}_2); d(W) = n-k+1\} = \emptyset \iff 1 < k < n-1$$

Proof. Assume that $w \in Gra\beta(k,n,\mathbb{F}_2)$, $d(W) = n-k+1$ and $k \geqslant 2$. Reorganize the order of axis (i.e. take a shuffle permutation a) such that $W \in fat(a)$. So we can choose a basis $(w_1, \ldots, w_k)$ such that last k coordinates formed identity $k \times k$ matrix. In order W has maximal Hamming distance all other entries should be non-zero (so they should be 1). In case $k \geqslant 2$ we can modify basis to $(w_1 + w_2, w_2, \ldots, w_k)$ and observe that now there are only two non-zero values in the first column $w_1 + w_2$, which contradicts with our assumption.

□

Generally

Lemma. *For every $W \in Gra\beta(k,n,\mathbb{F})$ there exists unique $\sigma \in \binom{[n]}{k}$ such as $W \in e(\sigma) \subset Gra\beta(n,k,\mathbb{F})$. Moreover $dist\ W \leq \sigma_1$.*

Now we can extend this observation to the any permutation of axis, i.e we can state:

Lemma. *For any permutation $a \in S_n$ and every $W \in Gra\beta(k,n,\mathbb{F})$ there exists unique $\sigma \in \binom{[n]}{k}$ such as $W \in e(a,\sigma) \subset Gra\beta(n,k,\mathbb{F})$. Moreover $dist\ W \leq (a \cdot \sigma)_1$.*

So we can denote this unique element by $(a,\sigma)(W)$ and prove follows

Theorem. *For every $W \in Gra\beta(k,n,\mathbb{F})$:*

$$dist\ W = \min_{a \in S_n} (a,\sigma)(W)_1$$

4. Looking for the best codes

Let $\sigma = (\sigma_1, \ldots, \sigma_k)$, where $1 \leqslant \sigma_1 < \ldots < \sigma_k \leqslant n$.

Denote $\sigma_k^n = \{\sigma; e(\sigma) \cup e(\tau) = \phi, \sigma \neq \tau\}$ and $\sigma(a) := (a_{\sigma_1}^{-1}, \ldots, a_{\sigma_k}^{-1})$ where a is a permutation of $[n]$.

Then

$$\bigcup_{\sigma_k^n} e(\sigma) = Grasß(k, n)$$

Of course $\sigma(id) = \sigma$ and

$$\forall_{a \in S_n} \bigcup_{\sigma_k^n} e(\sigma(a)) = Grasß(k, n)$$

Searching for the best codes we can narrow to the set:

$$FAT_{n,k} = \bigcap_{a \in S_n} fat(a)$$

Let $q = p^w$ and $|fat(a)| = \mathbb{F}_q$. When

$$|e(\sigma)| = q^{dim\ e(\sigma)} = q^{\sum_{i=1}^{k}(\sigma_i - i)}$$

Hence

$$|fat(a)| = q^{k(n-k)}$$

Now we are looking for $|\bigcap_{a \in S_n} fat(a)|$.

We have

$$G(k, n, q) = |Graß(k, n, \mathbb{F}_q)| =$$

$$\frac{(q^n - 1)(q^{n-1} - 1) \ldots (q^{b-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \ldots (q - 1)} \text{[3]}$$

and

$$fat(k, n, q) = |e(n - k + 1, \ldots, n)| = q^{k(n-k)}$$

Now using elementary inequalities we can estimate:

$$\frac{fat(k, n, q)}{G(k, n, q)} = q^{k(n-k)} \cdot \frac{(q^k - 1)(q^{k-1} - 1) \cdot \ldots \cdot (q - 1)}{(q^n - 1)(q^{n-1} - 1) \cdot \ldots \cdot (q^{b-k+1} - 1)}$$
$$\leqslant \ldots \leqslant$$
$$\exp(-\frac{1 - (\frac{1}{q^k} + \frac{1}{q^{n-k}} - \frac{1}{q^n})}{q - 1}).$$

Assume

$$q = p^w$$

from Gilbert-Varshamov [6] bound

$$q^{n-k} > \sum_{j=0}^{d-2} \binom{n-1}{j} (q-1)^j, \text{for } d \leqslant n-k+1$$

Let $d = n-k+2$

$$q^{n-k} \leqslant \sum_{j=0}^{n-k} \binom{n-1}{j} (q-1)^j$$

$m = n-k, \;\; 0 \leqslant m \leqslant n$

$$q^m \leqslant \sum_{j=0}^{m} \binom{n-1}{j} (q-1)^j$$

$$q^m = (1+(q-1))^m = \sum_{j=0}^{m} \binom{m}{j} (q-1)^j$$

Hence $0 < m \leqslant n-1$ and

$$\binom{a}{k} = \binom{a-1}{k} + \binom{a-1}{k-1} \Rightarrow \binom{a-1}{k} < \binom{a}{k}, \quad \text{for } \; a,k > 0$$

we get

$$\binom{m}{j} \leqslant \binom{n-1}{j}$$

So for $m = n$ we have

$$q^n = \sum_{j=0}^{n} \binom{n-1}{j} (q-1)^j \leqslant \sum_{j=0}^{n} \binom{n}{j} (q-1)^j + 0$$

It is easy to state, that

$$\lim_{q \to \infty} \frac{\sum_{j=0}^{n-k-1} \binom{n-1}{j} (q-1)^j}{q^{n-k}} \;=\; 0$$

From above we conclude that for sufficient large q

$$\frac{\sum_{j=0}^{n-k-1} \binom{n-1}{j} (q-1)^j}{q^{n-k}} \;<\; 1$$

Applying above inequality to *Gilbert-Varshamov [6] bound*, we get

Corollary. *For a fixed n, k and sufficient large q there exists linear code with size n dimension k over finite field $\mathbb{F}_q$ (q elements field) with arbitrary admissible minimal Hamming distance d, i.e.*

$$1 \leqslant d \leqslant n-k+1$$

.

5. Weight distribution

Now we consider family of codes of size 7 and dimension 4 over the smallest field $\mathbb{F}_q$ of characteristic not equal two, such that we achive maximum *Hamming distance*, i.e. $d = 4 = 7 - 4 + 1$. From the *Gilbert-Varshamov [6] bound*

$$q^{n-k} > \sum_{j=0}^{d-2} \binom{n-1}{j} (q-1)^j, \quad \text{for} \quad d \leqslant n-k+1$$

we see that q should be $q = 25 = 5^2$.

In order to construct the field $\mathbb{F}_{5^2}$ we use quadratic extension of prime field $\mathbb{F}_5$ by choosing irreducible polynomial

$$f = X^2 + X + 1 \in \mathbb{F}_5[X].$$

Then the ring of polynomials, truncated by ideal generated by f is isomorphic with desired field, i.e.

$$\mathbb{F}_5[X]/(f) \approx \mathbb{F}_{5^2}.$$

In order to iterate through all elements of field we choose generator g of multiplicative group of order $5^2 - 1 = 24$

$$g = \pi(2X + 3)$$

where π id natural projection:

$$\pi : \mathbb{F}_5[X] \longrightarrow \mathbb{F}_5[X]/(f) \approx \mathbb{F}_{5^2}.$$

With above data we choose applying „Monte Carlo" based algorithm distinct 103145 linear codes of size 7 and dimension 4 over the field $\mathbb{F}_{25} = \mathbb{F}_{5^2}$

For fixed linear code $C \subset \mathbb{F}^n$ of size n and dimension k over field $\mathbb{F}$ we introduce sequence $a_0, a_1, \ldots, a_n \in \mathbb{Z}$ by formula

$$a_j = \#\{\, v \in C \subset \mathbb{F}^n : \, |v| \, = j \,\}, \quad j = 0, 1, \ldots, n.$$

Obviously $a_0 = 1$.

Formal polynomial $f_C \in \mathbb{Z}[X]$ of degree n

$$f_C = \sum_{j=0}^{n} a_j X^j = a_0 + a_1 X + a_2 X^2 + \ldots + a_n X^n$$

codes

or in homogeneous form $\hat{f}_C \in \mathbb{Z}[X, Y], \;\; \hat{f}_C(X, Y) = Y^n \cdot f_C(\frac{X}{Y})$

$$\hat{f}_C = \sum_{j=0}^{n} a_j X^j Y^{n-j} = a_0 Y^n + a_1 X Y^{n-1} + a_2 X^2 Y^{n-2} + \ldots + a_n X^n.$$

we call *weight distribution polynomial*.

In our case for randomly chosen 103145 codes we get following distributions of code characteristics, first weight distribution. In order to shorten notation we write down only coefficients of weight polynomials and occurrence frequency.

For instance, when weight polynomial

$$1 + 0X + 0X^2 + 24X^3 + 744X^4 + 10728X^5 + 85584X^6 + 293544X^7 \ \in \ \mathbb{Z}[X]$$

occurred 43152 times, we write

```
[1 0 0 24 744 10728 85584 293544]:43152.
```

So we get sequence of 38 polynomials with decreasing frequencies:

```
[1 0 0 24 744 10728 85584 293544]:43152
[1 0 0 48 648 10872 85488 293568]:24935
[1 0 0 0 840 10584 85680 293520]:22142
[1 0 0 72 552 11016 85392 293592]:5128
[1 0 0 96 1056 9360 87096 293016]:2353
[1 0 0 120 960 9504 87000 293040]:1501
[1 0 24 0 600 11064 85320 293616]:1356
[1 0 24 48 1008 9552 86928 293064]:746
[1 0 24 24 504 11208 85224 293640]:686
[1 0 0 96 456 11160 85296 293616]:327
[1 0 24 72 912 9696 86832 293088]:225
[1 0 0 144 864 9648 86904 293064]:220
[1 0 0 240 2880 18024 63720 305760]:65
[1 0 24 48 408 11352 85128 293664]:62
[1 0 24 96 1416 8040 88536 292512]:62
[1 0 24 168 2928 18072 63648 305784]:35
[1 24 0 0 360 11664 84816 293760]:20
[1 0 24 96 816 9840 86736 293112]:19
[1 24 0 24 864 10008 86520 293184]:19
[1 0 0 264 2784 18168 63624 305784]:14
[1 0 0 192 1272 8136 88512 292512]:13
[1 0 48 0 960 9744 86760 293112]:13
[1 0 72 552 312 8136 89592 291960]:11
[1 0 24 120 1320 8184 88440 292536]:9
[1 0 48 48 1368 8232 88368 292560]:5
[1 0 0 168 768 9792 86808 293088]:4
[1 0 24 192 2832 18216 63552 305808]:4
[1 0 0 120 360 11304 85200 293640]:3
[1 0 48 24 864 9888 86664 293136]:3
```

```
[1 0 0 480 7920 76464 305760 0]:2
[1 0 48 96 2976 18120 63576 305808]:2
[1 24 0 48 1368 8352 88224 292608]:2
[1 24 0 96 2976 18240 63432 305856]:2
[1 0 24 144 1224 8328 88344 292560]:1
[1 0 24 240 2640 18504 63360 305856]:1
[1 0 72 576 216 8280 89496 291984]:1
[1 0 72 624 1824 18168 64704 305232]:1
[1 24 0 120 3480 16584 65136 305280]:1
```

The graph of above with natural ascending lexicographical order of polynomials is shown in Figure 1:

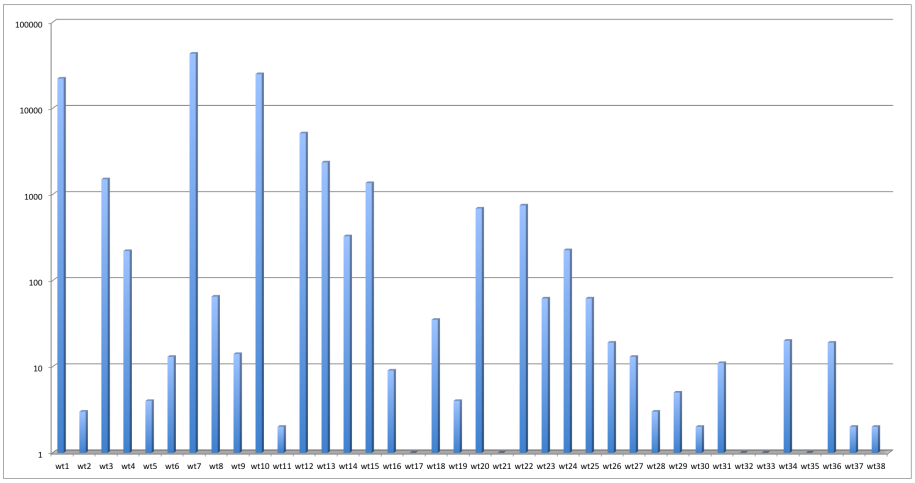

Figure 1. Hammings weights distribution (logarithmic scale).

Similarly we get distributions of types of *Schubert cells*

```
[4 5 6 7]:98806
[3 5 6 7]:4009
[3 4 6 7]:164
[2 5 6 7]:142
[2 4 6 7]:9
[1 5 6 7]:8
[3 4 5 7]:6
[2 3 6 7]:1
```

and *Hamming distances* (see Figure 2)

```
3:77717 (75.37%)
4:22142 (21.53%)
2:3242  (3.06%)
1:44    (0.04%)
```

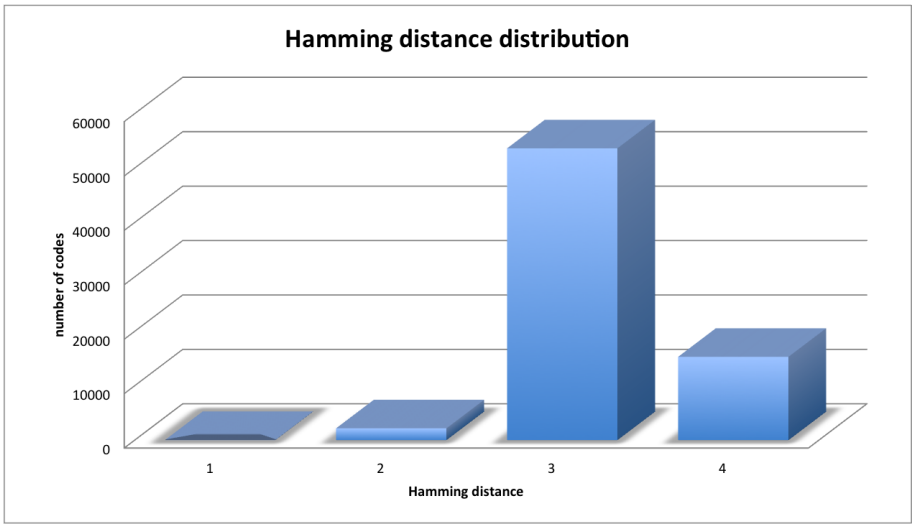

Figure 2. Hamminga distance distribution.

As we can seen in our case, we obtained 21.53% codes of the optimal Hamming dimension equal to 4.

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ROZKŁAD WAG HAMMINGA DLA PEWNEJ KLASY KODÓW LINIOWYCH

Streszczenie - Artykuł opisuje jakościowe własności pewnej klasy kodów liniowych o niskim wymiarze i kowymiarze, w szczególności zgodnym z oryginalnym kodem Hamminga. Kody te konstruowane są z wykorzystaniem algorytmu typu Monte Carlo. Rozważano kody czterowymiarowe zanurzone w siedmiowymia-rowej przestrzeni nad 25-elementowym ciałem $\mathbb{F}_{5^2}$. W pracy przedstawiono również analizę odległości Hamminga dla tych kodów.

Słowa kluczowe: Kody liniowe, rozmaitość Grassmana, odległość Hamminga