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THE SYNTHESES OF NUMERICAL METHODS OF COMPLEX ANALYSIS AND NUMERICAL-ANALYTICAL REPRESENTATIONS FOR SOLVING OF A CLASS OF NONLINEAR BOUNDARY VALUE PROBLEMS IN CURVILINEAR DOMAINS

Summary – The more efficient constructive approach to solving of nonlinear elliptic boundary value problems for doubly-connected curvilinear domains bounded by equipotential lines was developed on the basis of syntheses of the numerical methods complex analysis and numerical-analytical representations. Developed algorithm used to calculate the characteristic parameters of the filtering process in the shale layer and automatically solves the problem of choice of nodes and the construction of a dynamic grid, finding unknown parameters (the total flow, values speed, etc…).

Keywords: conformal (quasiconformal) mapping, method of summary representations, numerical-analytical representation, nonlinear boundary value problem, dynamic grid

1 Introduction

At present the constructive approach to modeling of quasiideal fields for curvilinear domains bounded by lines of flow and equipotential lines was developed based on methods of quasiconformal mappings and gradual fixation of characteristics process and environment characteristics (see eg. [1, 2]). This is the basis for the study of filtration processes, convection, mass transfer, diffusion in heterogeneous porous layers [3]. In [4-5] proposed the more efficient computing technology that combines numerical methods of complex analysis with numerical-analytical methods of summary representations [6-8]. This makes it possible for each step of iterative process approximating the desired nodes with considering all internal and boundary nodes. In this paper, the method of summary representations is generalized for case solving of model nonlinear problems that describes the stationary filtration process in curvilinear plane layers with conductivity that is the function of the field potential, by combining the numerical (finite-difference) and analytical methods (separation of variables, integral representations, etc. [9]).

2 Formulation of the problem

Consider the stationary filtration process in doubly-connected curvilinear domain G_z $(z = x + iy)$ bounded by closed contours – equipotential lines $L_* = \{z : f_*(x, y) = 0\}$ and $L^* = \{z : f^*(x, y) = 0\}$. As in [4], we make a imaginary incision Γ along the selected line of flow and get a simply-connected domain $G_z^{\Gamma} = G_z / \Gamma$.

We describe the filtering process as equation of motion $\vec{\upsilon} = \kappa(\varphi) \cdot grad\ \varphi$ (Darcy's law) and the continuity equation $div\, \vec{\upsilon} = 0$ [1], where $\vec{\upsilon} = \upsilon_x(x, y) + \mathrm{i}\, \upsilon_y(x, y)$ – filtration rate; $\kappa(\varphi)$ – the so-called fictitious conductivity coefficient, which characterizes the permeability of the medium, its susceptibility to deformation, density and viscosity of the substance to be filtered; φ – the potential of field, such that $\varphi\big|_{L_*} = \varphi_*$, $\varphi\big|_{L^*} = \varphi^*$, $-\infty < \varphi_* < \varphi^* < +\infty$.

Entering as in [1], the function flow $\psi = \psi(x, y)$ (quasiconformal adjoint to φ), we arrive to a more general problem on quasiconformal mapping $\omega = \omega(z) = \varphi(x, y) + i\,\psi(x, y)$ of the domain G_z^{Γ} on the rectangular domain of quasicomplex potential $G_\omega = \{\omega = \varphi + i\,\psi :$ $\varphi_* < \varphi < \varphi^*, \quad 0 < \psi < Q\}$ with unknown parameter (total flow) $Q = \oint\limits_{L_*} -\upsilon_y dx + \upsilon_x dy$:

$$\begin{cases} \dfrac{\partial \varphi}{\partial x} = \kappa(\varphi)\dfrac{\partial \psi}{\partial y}, \quad \kappa(\varphi)\dfrac{\partial \varphi}{\partial y} = -\dfrac{\partial \psi}{\partial x}, \quad (x,y) \in G_z^{\Gamma}; \\ \varphi\big|_{L_*} = \varphi_*, \quad \varphi\big|_{L^*} = \varphi^*, \quad \psi\big|_{L_0} = 0, \quad \psi\big|_{L^0} = \oint\limits_{L_*} -\dfrac{\partial \varphi}{\partial y}dx + \dfrac{\partial \varphi}{\partial x}dy \quad . \end{cases} \tag{1}$$

where L_0, L^0 – the lower and upper borders of the incision Γ.

The boundary value problem that inverse to (1) on quasiconformal mapping $z = z(\omega) = x(\varphi,\psi) + iy(\varphi,\psi)$ of the domain G_ω to G_z^{Γ} with unknown flow Q and with the imaginary incision Γ looks like:

$$\begin{cases} \kappa(\varphi)\dfrac{\partial x}{\partial \varphi} = \dfrac{\partial y}{\partial \psi}, \quad \dfrac{\partial y}{\partial \varphi} = -\kappa(\varphi)\dfrac{\partial x}{\partial \psi}, \quad (\varphi,\psi) \in G_\omega \\ f_*\left(x(\varphi_*,\psi), y(\varphi_*,\psi)\right) = 0, \quad f^*\left(x(\varphi^*,\psi), y(\varphi^*,\psi)\right) = 0, \quad 0 \le \psi \le Q \\ x(\varphi,0) = x(\varphi,Q), \quad y(\varphi,0) = y(\varphi,Q), \quad \varphi_* \le \varphi \le \varphi^* \\ \lim\limits_{\psi \to 0+0} \dfrac{\partial x(\varphi,\psi)}{\partial \psi} = \lim\limits_{\psi \to Q-0} \dfrac{\partial x(\varphi,\psi)}{\partial \psi}, \quad \lim\limits_{\psi \to 0+0} \dfrac{\partial y(\varphi,\psi)}{\partial \psi} = \lim\limits_{\psi \to Q-0} \dfrac{\partial y(\varphi,\psi)}{\partial \psi} \end{cases} \tag{2}$$

and is equivalent to the problem [4]:

$$\begin{cases} \dfrac{\partial}{\partial \varphi}\left(\dfrac{1}{\kappa(\varphi)}\dfrac{\partial x}{\partial \varphi}\right) + \dfrac{\partial}{\partial \psi}\left(\kappa(\varphi)\dfrac{\partial x}{\partial \psi}\right) = 0, \\ \dfrac{\partial}{\partial \varphi}\left(\dfrac{1}{\kappa(\varphi)}\dfrac{\partial y}{\partial \varphi}\right) + \dfrac{\partial}{\partial \psi}\left(\kappa(\varphi)\dfrac{\partial y}{\partial \psi}\right) = 0, \quad (\varphi,\psi) \in G_\omega \\ f_*\left(x(\varphi_*,\psi), y(\varphi_*,\psi)\right) = 0, \quad f^*\left(x(\varphi^*,\psi), y(\varphi^*,\psi)\right) = 0, \quad 0 \le \psi \le Q \\ x(\varphi,0) = x(\varphi,Q), \quad y(\varphi,0) = y(\varphi,Q), \quad \varphi_* \le \varphi \le \varphi^* \\ \left.\left(\dfrac{\partial x}{\partial \varphi}\cdot\dfrac{\partial f_*}{\partial y} - \dfrac{\partial y}{\partial \varphi}\cdot\dfrac{\partial f_*}{\partial x}\right)\right|_{\varphi=\varphi_*} = 0, \quad \left.\left(\dfrac{\partial x}{\partial \varphi}\cdot\dfrac{\partial f^*}{\partial y} - \dfrac{\partial y}{\partial \varphi}\cdot\dfrac{\partial f^*}{\partial x}\right)\right|_{\varphi=\varphi^*} = 0 \\ Q = \oint\limits_{L_*} \dfrac{\kappa(\varphi)}{J}\dfrac{\partial x}{\partial \psi}dx + \dfrac{\kappa(\varphi)}{J}\dfrac{\partial y}{\partial \psi}dy, \quad J = \dfrac{\partial x}{\partial \varphi}\dfrac{\partial y}{\partial \psi} - \dfrac{\partial x}{\partial \psi}\dfrac{\partial y}{\partial \varphi} \end{cases} \tag{3}$$

3 Numerical-analytical representations of solutions

Numerical-analytical representations of solutions of the nonlinear problem (3) are constructed on idea gradual fixation of some parameters and a combination of the numerical (finite-difference) and analytical methods (separation of variables, integral representations, etc…) which are generalizations of methods of summary representations.

In the domain of quasicomplex potential we building the uniform orthogonal grid and we replace G_ω over the grid domain $G_\omega^\gamma = \{(\varphi_i, \psi_j)$: $\varphi_i = \varphi_* + \Delta_\varphi \cdot i$, $i = \overline{0, m+1}$; $\psi_j = \Delta_\psi \cdot j$, $j = \overline{0, n+1}$; $\Delta_\varphi = \dfrac{\varphi^* - \varphi_*}{m+1}$, $\Delta_\psi = \dfrac{Q}{n+1}$, $\gamma = \Delta_\varphi / \Delta_\psi$, m, $n \in \mathbf{N}\}$, and boundary conditions, the conditions of periodicity on incision and conditions of orthogonality lines dynamic grid to the borders the physical domain – finite-difference analogs [4]:

$$\begin{cases} f_*\left(x_{0,j}, y_{0,j}\right) = 0, \quad f^*\left(x_{m+1,j}, y_{m+1,j}\right) = 0, \quad j = \overline{0, n+1} \\ x_{i,0} = x_{i,n+1}, \quad y_{i,0} = y_{i,n+1}, \quad i = \overline{0, m+1} \end{cases} \tag{4}$$

$$\begin{cases} x_{i,j} = \alpha(x_{i+1,j} + x_{i-1,j}) + \beta(x_{i,j+1} + x_{i,j-1}) + 0.05\left(x_{i-1,j-1} + x_{i+1,j-1} + x_{i-1,j+1} + x_{i+1,j+1}\right) \\ y_{i,j} = \alpha(y_{i+1,j} + y_{i-1,j}) + \beta(y_{i,j+1} + y_{i,j-1}) + 0.05\left(y_{i-1,j-1} + y_{i+1,j-1} + y_{i-1,j+1} + y_{i+1,j+1}\right) \\ i = \overline{1, m}, \quad j = \overline{0, n+1}, \quad \alpha = \dfrac{3\Delta_\varphi{}^2 \Delta_\psi{}^4}{5(\Delta_\varphi{}^2 \Delta_\psi{}^4 + \Delta_\varphi{}^4 \Delta_\psi{}^2)} - 0.1, \quad \beta = \dfrac{3\Delta_\varphi{}^4 \Delta_\psi{}^2}{5(\Delta_\varphi{}^2 \Delta_\psi{}^4 + \Delta_\varphi{}^4 \Delta_\psi{}^2)} - 0.1 \end{cases} \tag{5}$$

$$\begin{cases} \left(4x_{1,j} - 3x_{0,j} - x_{2,j}\right)\left(x_{0,j+1} - x_{0,j-1}\right) + \left(4y_{1,j} - 3y_{0,j} - y_{2,j}\right)\left(y_{0,j+1} - y_{0,j-1}\right) = 0 \\ \left(3x_{m,j} + x_{m-2,j} - 4x_{m-1,j}\right)\left(x_{m,j+1} - x_{m,j-1}\right) + \left(3y_{m,j} + y_{m-2,j} - 4y_{m-1,j}\right)\left(y_{m,j+1} - y_{m,j-1}\right) = 0 \\ \left(4x_{i,1} - 3x_{i,0} - x_{i,2}\right)\left(x_{i+1,0} - x_{i-1,0}\right) + \left(4y_{i,1} - 3y_{i,0} - y_{i,2}\right)\left(y_{i+1,0} - y_{i-1,0}\right) = 0 \\ \left(3x_{i,n} + x_{i,n-2} - 4x_{i,n-1}\right)\left(x_{i+1,n} - x_{i-1,n}\right) + \left(3y_{i,n} + y_{i,n-2} - 4y_{i,n-1}\right)\left(y_{i+1,n} - y_{i-1,n}\right) = 0 \\ i = \overline{1, m}, \quad j = \overline{1, n} \end{cases} \tag{6}$$

where $x_{i,j} = x(\varphi_i, \psi_j)$, $y_{i,j} = y(\varphi_i, \psi_j)$.

For fixed (given) initial values of the unknown γ (or desired flow Q) and functions x and y in the boundary nodes of the grid domain $x_{0,j}$, $y_{0,j}$, $x_{m+1,j}$, $y_{m+1,j}$, $x_{i,n+1}$, $y_{i,n+1}$, $x_{i,0}$, $y_{i,0}$ (considering the boundary conditions (4)), approximation of functions x and y in the internal nodes of the grid domain we find as solutions to these problems:

$$\begin{cases} \mathrm{L}x = 0, \quad (\varphi,\psi) \in G_\omega \\ x(\varphi_*,\psi) = \bar{x}_1(\psi), \quad x(\varphi^*,\psi) = \bar{x}_2(\psi), \quad x(\varphi,0) = \bar{x}_3(\varphi), \quad x(\varphi,Q) = \bar{x}_4(\varphi) \end{cases} \tag{7}$$

$$\begin{cases} \mathrm{L}y = 0, \quad (\varphi,\psi) \in G_\omega \\ y(\varphi_*,\psi) = \bar{y}_1(\psi), \quad y(\varphi^*,\psi) = \bar{y}_2(\psi), \quad y(\varphi,0) = \bar{y}_3(\varphi), \quad y(\varphi,Q) = \bar{y}_4(\varphi) \end{cases} \tag{8}$$

where $\mathrm{L}u \equiv \dfrac{\partial}{\partial\varphi}\left(\dfrac{1}{\kappa(\varphi)}\dfrac{\partial u(\varphi,\psi)}{\partial\varphi}\right) + \dfrac{\partial}{\partial\psi}\left(\kappa(\varphi)\dfrac{\partial u(\varphi,\psi)}{\partial\psi}\right), \quad \bar{x}_1(\psi_j) = x_{0,j}$,
$\bar{x}_2(\psi_j) = x_{m+1,j}$, $\bar{y}_1(\psi_j) = y_{0,j}$, $\bar{y}_2(\psi_j) = y_{m+1,j}$, $j = \overline{0,n+1}$,
$\bar{x}_3(\varphi_i) = x_{i,0}$, $\bar{x}_4(\varphi_i) = x_{i,n+1}$, $\bar{y}_3(\varphi_i) = y_{i,0}$, $\bar{y}_4(\varphi_i) = y_{i,n+1}$, $i = \overline{0,m+1}$
(here and later, $\bar{u}(\varphi,\psi)$, $(\varphi,\psi) \in G_\omega$ is denote the analytic continuation of grid function $u_{i,j} = u(\varphi_i,\psi_j)$, $(\varphi_i,\psi_j) \in G_\omega^\gamma$).

The solution of the problem (7) we searched as:

$$x(\varphi,\psi) = \bar{u}(\varphi,\psi) + v(\varphi,\psi)$$

where $$\bar{u}(\varphi,\psi) = \bar{x}_3(\varphi) + \frac{\psi}{Q}\left(\bar{x}_4(\varphi) - \bar{x}_3(\varphi)\right)$$

or in nodes of grid $u_{i,j} = \dfrac{n+1-j}{n+1}x_{i,0} + \dfrac{j}{n+1}x_{i,n+1}$, $i = \overline{0,m+1}$, $j = \overline{0,n+1}$
and the function $v(\varphi,\psi)$ is a solution of the following boundary value problem:

$$\begin{cases} \mathrm{L}v = \bar{F}(\varphi,\psi), \quad (\varphi,\psi) \in G_\omega, \\ v(\varphi_*,\psi) = \bar{v}_1(\psi), \quad v(\varphi^*,\psi) = \bar{v}_2(\psi), \quad v(\varphi,0) = 0, \quad v(\varphi,Q) = 0, \end{cases} \tag{9}$$

where
$\bar{F}(\varphi,\psi) = -\mathrm{L}\bar{u} = -\dfrac{\partial}{\partial\varphi}\left(\dfrac{1}{\kappa(\varphi)}\dfrac{\partial\bar{u}(\varphi,\psi)}{\partial\varphi}\right), \bar{v}_1(\psi) = \bar{x}_1(\psi) - \bar{u}(\varphi_*,\psi)$,
$\bar{v}_2(\psi) = \bar{x}_2(\psi) - \bar{u}(\varphi^*,\psi)$.

Using the method separation of variables we find solution of the problem (9) as a series:

$$v(\varphi,\psi)=\frac{2}{Q}\sum_{p=1}^{\infty}\Phi_p(\varphi)\sin\frac{\pi p}{Q}\psi \tag{10}$$

where functions $\Phi_p(\varphi)$ are the solutions of the problem:

$$\begin{cases}\left(\dfrac{1}{\kappa(\varphi)}\Phi'_p(\varphi)\right)' -\lambda_p\kappa(\varphi)\Phi_p(\varphi)=\overline{F}_p(\varphi)\\ \Phi_p(\varphi_*)=\overline{v}_{1p},\quad \Phi_p(\varphi^*)=\overline{v}_{2p}\end{cases} \tag{11}$$

$$\lambda_p=\left(\frac{\pi p}{Q}\right)^2,\ \overline{F}_p(\varphi)=\frac{2}{Q}\int_0^Q\overline{F}(\varphi,\xi)\sin\frac{\pi p}{Q}\xi d\xi\,,\ \overline{v}_{1p}=\frac{2}{Q}\int_0^Q\overline{v}_1(\xi)\sin\frac{\pi p}{Q}\xi d\xi\,,$$

$$\overline{v}_{2p}=\frac{2}{Q}\int_0^Q\overline{v}_2(\xi)\sin\frac{\pi p}{Q}\xi d\xi\,.$$

Finite-difference analog of (11) we construct by the method of the balance (integro-interpolation method), which provides second-order accuracy, convergence and stability of the difference problem [10]:

$$\begin{cases}\Phi_{p,i+1}-\left(1+\alpha_i+\lambda_p\beta_i\right)\Phi_{p,i}+\alpha_i\Phi_{p,i-1}=F^*_{p,i},\quad i=\overline{1,m}\\ \Phi_{p,0}=\overline{v}_{1p},\quad \Phi_{p,m+1}=\overline{v}_{2p}\end{cases} \tag{12}$$

where $\Phi_{p,i}=\Phi_p(\varphi_i)$, $F_{p,i}=\overline{F}_p(\varphi_i)$, $\kappa_i=\kappa(\varphi_i)$,

$\kappa_{i+1/2}=\dfrac{1}{\Delta_\varphi}\displaystyle\int_{\varphi_i}^{\varphi_{i+1}}\kappa(\xi)d\xi$, $\kappa_{i-1/2}=\dfrac{1}{\Delta_\varphi}\displaystyle\int_{\varphi_{i-1}}^{\varphi_1}\kappa(\xi)d\xi$, $\alpha_i=\dfrac{\kappa_{i-1/2}}{\kappa_{i+1/2}}$,

$\beta_i=\dfrac{\kappa_i}{\kappa_{i+1/2}}$, $F^*_{p,i}=\dfrac{F_{p,i}}{\kappa_{i+1/2}}$.

The solution (12) is represented as [10]:

$$\Phi_{p,i}=A_p\mu_{p,i}+B_p\nu_{p,i}+\mathrm{G}_{p,i}$$

where A_p, B_p – are constants determined from the boundary conditions; $\mu_{p,i},\nu_{p,i}$ – linearly-independent solutions of the homogeneous system of equations that can be derived from the recurrence relations:

$$\mu_{p,0}=0,\quad \mu_{p,1}=1,\quad \mu_{p,i+1}=\left(1+\alpha_i+\lambda_p\beta_i\right)\mu_{p,i}-\alpha_i\mu_{p,i-1}$$
$$\nu_{p,0}=1,\quad \nu_{p,1}=0,\quad \nu_{p,i+1}=\left(1+\alpha_i+\lambda_p\beta_i\right)\nu_{p,i}-\alpha_i\nu_{p,i-1}$$

$\mathrm{G}_{p,i}$ – the partial solution of system inhomogeneous equations with zero boundary conditions:

$$\mathrm{G}_{p,i}=\sum_{q=1}^{i-1}\frac{\mu_{p,i}\nu_{p,q}-\mu_{p,q}\nu_{p,i}}{\mu_{p,q-1}\nu_{p,q}-\mu_{p,q}\nu_{p,q-1}}F^*_{p,q}\,.$$

The values $F(\varphi,\psi)$, v_{1p}, v_{2p} in the grid are determined by formulas:

$$F_{i,j}=F(\varphi_i,\psi_j)=-\frac{1}{\Delta_\varphi}\left(\frac{1}{\kappa_{i+1/2}}\frac{u_{i+1,j}-u_{i,j}}{\Delta_\varphi}-\frac{1}{\kappa_{i-1/2}}\frac{u_{i,j}-u_{i-1,j}}{\Delta_\varphi}\right)=$$
$$=-\left(\frac{(n+1-j)x_{i+1/2,0}+jx_{i+1/2,n+1}}{(n+1)\Delta_\varphi^2\kappa_{i+1/2}}-\frac{(n+1-j)x_{i-1/2,0}+jx_{i-1/2,n+1}}{(n+1)\Delta_\varphi^2\kappa_{i-1/2}}\right)$$
$$i=\overline{1,m},\ j=\overline{1,n}$$
$$F^*_{p,i}=\frac{1}{\kappa_{i+1/2}}\sum_{j=1}^{n}P_{j,p}F_{i,j}=-\frac{\kappa_{i-1/2}\left(x_{i+1/2,0}+x_{i+1/2,n+1}\right)-\kappa_{i+1/2}\left(x_{i-1/2,0}+x_{i-1/2,n+1}\right)}{(n+1)\Delta_\varphi^2\kappa_{i+1/2}^2\kappa_{i-1/2}}\sum_{j=1}^{n}jP_{j,p}\ ,$$

$$v_{1,p}=\sum_{j=1}^{n}x_{0,j}P_{j,p}-\left(x_{0,0}+x_{0,n+1}\right)\sum_{j=1}^{n}jP_{j,p}$$
$$v_{2,p}=\sum_{j=1}^{n}x_{m+1,j}P_{j,p}-\left(x_{m+1,0}+x_{m+1,n+1}\right)\sum_{j=1}^{n}jP_{j,p}$$

where $P=\left[P_{j,k}\right]_{j,k=1}^{n}=\left[\sqrt{\frac{2}{n+1}}\sin\frac{\pi kj}{n+1}\right]_{j,k=1}^{n}$ – square symmetric matrix P-transformations [6-8].

Then the general solution of the problem (9) in the nodes of grid domain G_ω^γ ($v_{i,j}=v(\varphi_i,\psi_j)$, $i=\overline{0,m+1}$, $j=\overline{1,n}$) has the form:

$$v_{i,j}=\sum_{p=1}^{n}P_{j,p}\left(A_p\mu_{p,i}+B_p\nu_{p,i}-\sum_{q=1}^{i-1}\frac{\mu_{p,i}\nu_{p,q}-\mu_{p,q}\nu_{p,i}}{\mu_{p,q-1}\nu_{p,q}-\mu_{p,q}\nu_{p,q-1}}\times\right.$$
$$\left.\times\frac{\kappa_{q-1/2}\left(x_{q+1/2,0}+x_{q+1/2,n+1}\right)-\kappa_{q+1/2}\left(x_{q-1/2,0}+x_{q-1/2,n+1}\right)}{(n+1)\Delta_{\varphi}^2\kappa_{q+1/2}^2\kappa_{q-1/2}}\sum_{j=1}^{n}jP_{j,p}\right)$$

and a general solution of the problem (7):

$$x_{i,j}=\frac{n+1-j}{n+1}x_{i,0}+\frac{j}{n+1}x_{i,n+1}+\sum_{p=1}^{n}P_{j,p}\left(A_p\mu_{p,i}+B_p\nu_{p,i}-\sum_{q=1}^{i-1}\frac{\mu_{p,i}\nu_{p,q}-\mu_{p,q}\nu_{p,i}}{\mu_{p,q-1}\nu_{p,q}-\mu_{p,q}\nu_{p,q-1}}\times\right.$$
$$\left.\times\frac{\kappa_{q-1/2}\left(x_{q+1/2,0}+x_{q+1/2,n+1}\right)-\kappa_{q+1/2}\left(x_{q-1/2,0}+x_{q-1/2,n+1}\right)}{(n+1)\Delta_{\varphi}^2\kappa_{q+1/2}^2\kappa_{q-1/2}}\sum_{j=1}^{n}jP_{j,p}\right) \tag{13}$$

Similarly, the solution of the problem (8) is found as:

$$y_{i,j}=\frac{n+1-j}{n+1}y_{i,0}+\frac{j}{n+1}y_{i,n+1}+\sum_{p=1}^{n}P_{j,p}\left(C_p\mu_{p,i}+D_p\nu_{p,i}-\sum_{q=1}^{i-1}\frac{\mu_{p,i}\nu_{p,q}-\mu_{p,q}\nu_{p,i}}{\mu_{p,q-1}\nu_{p,q}-\mu_{p,q}\nu_{p,q-1}}\times\right.$$
$$\left.\times\frac{\kappa_{q-1/2}\left(y_{q+1/2,0}+y_{q+1/2,n+1}\right)-\kappa_{q+1/2}\left(y_{q-1/2,0}+y_{q-1/2,n+1}\right)}{(n+1)\Delta_{\varphi}^2\kappa_{q+1/2}^2\kappa_{q-1/2}}\sum_{j=1}^{n}jP_{j,p}\right). \tag{14}$$

4 Algorithm

The algorithm for solving the problem (3) is the same as [4-5], and in general can be described as follows. We give the number of nodes $m\times n$ partitioning of the grid domain G_{ω}^{γ}, a parameter ε that characterizes the accuracy of the approximation solution of the difference problems and desired degree of quasiconformal mapping δ_*, zero approximation of unknown values γ (or unknown flow Q), the initial approximation of values of functions x and y in the boundary nodes (coordinates of the boundary nodes of dynamic grid) so that the conditions (4) and we calculate by formulas (13)-(14) the initial approximations of functions x and y in the internal nodes (coordinates of the internal nodes of dynamic grid). We find γ and Q by formulas:

$$\gamma = \frac{1}{(m+1)(n+1)} \sum_{i,j=0}^{m,n} \gamma_{i,j}\,, \qquad Q = \Delta_\varphi \cdot \frac{n+1}{\gamma} \tag{15}$$

where

$$\gamma_{i,j} = \frac{\sqrt{\left(x_{i+1,j} - x_{i,j}\right)^2 + \left(y_{i+1,j} - y_{i,j}\right)^2} + \sqrt{\left(x_{i+1,j+1} - x_{i,j+1}\right)^2 + \left(y_{i+1,j+1} - y_{i,j+1}\right)^2}}{\sqrt{\left(x_{i,j+1} - x_{i,j}\right)^2 + \left(y_{i,j+1} - y_{i,j}\right)^2} + \sqrt{\left(x_{i+1,j+1} - x_{i+1,j}\right)^2 + \left(y_{i+1,j+1} - y_{i+1,j}\right)^2}}$$

Then we refine the coordinates of the boundary nodes by (4)-(6) and compute a new approximation of coordinates of the internal nodes by the formulas (13)-(14), and find γ and Q by (15).

At the end of each iteration we check the conditions stabilizing the coordinates of boundary nodes, if the magnitude of the displacement of nodes on the boundary for ongoing k-th iteration $S = \max_{i,j} \sqrt{\left(x_{i,j}^{(k)} - x_{i,j}^{(k-1)}\right)^2 + \left(y_{i,j}^{(k)} - y_{i,j}^{(k-1)}\right)^2}$ is greater than ε, then we repeat the recalculate parameters of the problem. Otherwise, we stop the iteration process and we estimate the degree of the quasiconformality $\delta = \sqrt{\delta_1^2 + \delta_2^2}$ of the resulting mapping the domain of complex potential on the physical domain, where δ_1, δ_2 - residual approximations Cauchy-Riemann conditions:

$$\delta_1 = \max_{i,j=2}^{m-1,n-1} \left| \kappa_i (x_{i+1,j} - x_{i-1,j}) - \gamma (y_{i,j+1} - y_{i,j-1}) \right|,$$

$$\delta_2 = \max_{i,j=2}^{m-1,n-1} \left| (y_{i+1,j} - y_{i-1,j}) + \kappa_i \gamma (x_{i,j+1} - x_{i,j-1}) \right|.$$

If $\delta \le \delta_*$, then we believe that the problem is solved with the required precision, otherwise, we increase the number of nodes of the partition domain or we change the ratio between m and n (if possible, values m and n should be set so that $\gamma \approx 1$; geometrically this means that almost square grid) and we repeat steps of the algorithm.

5 The partials cases

The partials cases of functions $\kappa(\varphi)$ can significantly simplify the views of the formulas (13), (14) or can get a solution for the problems

(7), (8) in an analytical form, which increases the efficiency of this algorithm.

Note that the formulas (13), (14) are generalization of the classical formulas of summary representations [6-8] to the case problems (7), (8): if put $\kappa(\varphi)=1$, then the formula (13) ((14) – similar) will form:

$$x_{i,j}=\sum_{p=1}^{n}P_{j,p}\left(A_p\mu_p^i+B_p\nu_p^i-\gamma^2\sum_{q=1}^{i-1}\frac{\mu_p^{i-q}-\nu_p^{i-q}}{\mu_p-\nu_p}\left(P_{1,p}x_{q,0}+P_{n,p}x_{q,n+1}\right)\right) \tag{16}$$

where μ_p^i,ν_p^i – solutions of system homogeneous equations (12), defined as the roots of the characteristic equation $r^2-2\eta_p r+1=0$: $\mu_p=\eta_p-\sqrt{\eta_p^2-1}$, $\nu_p=\eta_p-\sqrt{\eta_p^2-1}$, $\eta_p=1+\gamma^2\lambda_p$.

In the case $\kappa(\varphi)=e^{\sigma\varphi}$ the formula (13) also will look similar to the classic formula of summary representations (16) with some modifications:

$$x_{i,j}=\sum_{p=1}^{n}P_{j,p}\left(A_p\mu_p^i+B_p\nu_p^i-\frac{\gamma^2\sigma\Delta_\varphi}{e^{\sigma\Delta_\varphi}-1}\sum_{q=1}^{i-1}\frac{\mu_p^{i-q}-\nu_p^{i-q}}{\mu_p-\nu_p}\left(P_{1,p}x_{q,0}+P_{n,p}x_{q,n+1}\right)\right)$$

where $\mu_p=\eta_p+\sqrt{\eta_p^2-e^{\sigma\Delta_\varphi}}$, $\nu_p=\eta_p-\sqrt{\eta_p^2-e^{\sigma\Delta_\varphi}}$, $\eta_p=\dfrac{e^{\sigma\Delta_\varphi}}{2}+\gamma^2\lambda_p$.

In the case $\kappa(\varphi)=\varphi^k$, $k=\pm1,\pm2,\ldots$ in (11) we have Bessel differential equation, it's solutions can be written analytically using cyclic functions [10]:

$$\Phi_p(\varphi)=\varphi^{\frac{k+1}{2}}J_{\pm 1/2}\left(\frac{\lambda_p\varphi^{k+1}}{k+1}\right)$$

where $J_{\pm 1/2}$ – Bessel functions of the first kind, $J_{1/2}(z)=\sqrt{\dfrac{2}{\pi}}\dfrac{\sin z}{\sqrt{z}}$, $J_{-1/2}(z)=\sqrt{\dfrac{2}{\pi}}\dfrac{\cos z}{\sqrt{z}}$.

6 Conclusion

Thus, we constructed the effective approach to solve a wide class of nonlinear boundary value problems that are models stationary filtration processes in porous layers in which the coefficient of conductivity is a function of the potential of field, based on the syntheses of numerical complex analysis and numerical-analytical methods. Our method avoids the accumulation of computational errors and is convenient for computer implementation. Prospect research is to extend the proposed approach to forecasting processes described in [3].

7 References

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SYNTEZA NUMERYCZNYCH METOD ANALIZY ZESPOLONEJ I NUMERYCZNO-ANALITYCZNEJ REPREZENTACJI W ROZWIĄZANIACH PEWNEJ KLASY NIELINIOWYCH PROBLEMÓW BRZEGOWYCH W ZAKRZYWIONYM OBSZARZE

Streszczenie: W pracy przedstawiono efektywne podejście do rozwiązywania nieliniowych eliptycznych problemów brzegowych w dwu-spójnym krzywoliniowym obszarze ograniczonym liniami ekwipotencjalnymi, wykorzystując połączenie numerycznych metod analizy zespolonej i reprezentacji numeryczno-analitycznej. Opracowany algorytm wykorzystywano do obliczania charakterystycznych parametrów procesu filtracji w warstwie łupków, automatycznie rozwiązując problem wyboru węzłów i budowy dynamicznej siatki, w celu wyznaczenia nieznanych parametrów problemu (np. całkowitego przepływu, wartości prędkości, itp.).

Słowa kluczowe: odwzorowanie konforemne, metoda sumarycznej reprezentacji, reprezentacja numeryczno-analityczna, nieliniowe problemy brzegowe, siatka dynamiczna